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المدرسة العليا للاقتصاد وهران

ORAN GRADUATE SCHOOL OF ECONOMICS

PEDAGOGICAL HANDOUT

IN MATHEMATICAL ANALYSIS

Courses and application exercises intended for first-year preparatory students

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Preface

The purpose of this handout is to present, in a clear and elementary manner, the fundamental mathematical techniques and results in analysis that first-year students in management and economics are expected to master. The intention is not to prove the stated theorems or propositions, but rather to elucidate their applications and the underlying computational rules.

A substantial part of the material presented in this handout is drawn from the lectures I have delivered at the Higher School of Economics in Oran.

This handout is composed of six chapters. Each chapter includes a series of exercises, most of which are accompanied by detailed solutions, allowing students to assess and consolidate their understanding.

I have endeavored to design a resource that can be used not only by students in mathematics, but also by those in economics and management, in the hope of contributing to the development of Algerian academic documentation.

Contents

Chapter 1

Fundamentals of Functions

1.1 Introduction to Real Numbers

1. We denote by $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ the set of natural numbers, and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ the set of positive natural numbers.

2. We denote by $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, +1, +2, +3, \ldots\}$ the set of integers, and by $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ the set of nonzero integers.

3. We define the set of rational numbers by $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}^*\}$.
   Rational numbers can be expressed as decimal fractions that are either *terminating* or *repeating*.

Example 1.1.1. $\frac{5}{4} = 1,25 \in \mathbb{Q}$; $\frac{10}{-3} = -3,33... \in \mathbb{Q}$; $\frac{5}{7} = 0,\underbrace{71425}\underbrace{71425}... \in \mathbb{Q}$; $7,25123123123..... \in \mathbb{Q}$.

Clearly, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$.

Definition 1.1.1. *Numbers represented by non-terminating, non-repeating decimal fractions are referred to as* ***irrational numbers****.*

Example 1.1.2. $\sqrt{2} = 1,414213... \notin \mathbb{Q}$; $\sqrt{7} = 2.645751311... \notin \mathbb{Q}$; $\pi = 3,141592... \notin \mathbb{Q}$.

Definition 1.1.2. *The set of real numbers, denoted by $\mathbb{R}$, is composed of both rational and irrational numbers.*

Thus, we have reached the following inclusion: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

1.2 Intervals in $\mathbb{R}$

Let a and b be two real numbers such that $a < b$. The types of intervals are as follows:

1. $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ (closed interval).
2. $]a, b[= \{x \in \mathbb{R} : a < x < b\}$ (open interval).
3. $]a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ (half-open interval).
4. $[a, b[= \{x \in \mathbb{R} : a \leq x < b\}$ (half-open interval).
5. $]a, +\infty[= \{x \in \mathbb{R} : x > a\}$ (open interval).
6. $[a, +\infty[= \{x \in \mathbb{R} : x \geq a\}$ (closed interval).
7. $]-\infty, b[= \{x \in \mathbb{R} : x < b\}$ (open interval).
8. $]-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$ (closed interval).
9. $]-\infty, +\infty[= \mathbb{R}$ (open interval).

1.3 Special Products (Notable Identities)

The most well-known identities are: $\forall (x, y) \in \mathbb{R}^2$

- $(x + y)^2 = x^2 + 2xy + y^2$.
- $(x - y)^2 = x^2 - 2xy + y^2$.
- $x^2 - y^2 = (x - y)(x + y)$.

To solve the quadratic equation $ax^2+bx+c = 0$, we compute the discriminant $\Delta = b^2-4ac$:

- If $\Delta > 0$, we have two distinct solutions $x_1 = \dfrac{-b - \sqrt{\Delta}}{2a}$ and $x_2 = \dfrac{-b + \sqrt{\Delta}}{2a}$.
- If $\Delta = 0$, we have a double root given by $x' = x'' = \dfrac{-b}{2a}$.
- If $\Delta < 0$, the equation has no real solution.

1.4 Some Basic Rules

1. $a \geq 0 \Leftrightarrow -a \leq 0$.

2. $a \geq b \Leftrightarrow -a \leq -b$.

3. $a \geq 0$ and $b \leq 0 \Rightarrow ab \leq 0$.

4. $a \leq 0$ and $b \leq 0 \Rightarrow ab \geq 0$.

5. $a \geq b \Leftrightarrow a - b \geq 0$.

6. If $b \geq 0$, then $\forall a \in \mathbb{R},\ a - b \leq a \leq a + b$.

7. $a > b$ et $c > 0 \Rightarrow ac > bc$ *(multiplying by c)*.

8. $a > 0 \Rightarrow \dfrac{1}{a} > 0$.

9. $0 < a < b \Rightarrow 0 < \dfrac{1}{b} < \dfrac{1}{a}$.

10. If $a \geq 0$ and $b \geq 0$, then $a^2 \leq b^2 \Leftrightarrow a \leq b$.

11. $a \leq b$ and $c \leq d \Rightarrow a + c \leq b + d$.

12. $\dfrac{a}{b} = \dfrac{a'}{b'} \Leftrightarrow ab' = a'b,\ \forall b, b' \neq 0$.

1.5 Real-Valued Functions

Definition 1.5.1. *Let E and F be subsets of $\mathbb{R}$.*

1. *We say that f is a real-valued function from E to F if it assigns to each element x of E an element y of F, denoted by $f(x)$, and we write*

$$\begin{aligned} f : E &\longrightarrow F \\ x &\longmapsto f(x). \end{aligned}$$

$f(x)$ is called the image of x under f.
E is called the domain of f, and F is called the codomain.

2. *The set of all $x \in E$ that have an image $y \in F$ is called the domain of definition of f, denoted by:*

$$D_f = \{x \in E \mid \exists\, y \in F,\ y = f(x)\}.$$

Example 1.5.1.

$$\begin{aligned} f : E &\longrightarrow E \\ x &\longmapsto f(x) = x = I_E, \end{aligned}$$

is called the identity function.

Example 1.5.2.

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \frac{x}{x-1} \end{aligned}$$

The expression above defines a real-valued function. However, before using f, we must specify the set of all real numbers x for which $f(x)$ is well-defined. Since the denominator $x-1$ must not be zero, we exclude $x = 1$ from the domain.
Hence, the domain of f is given by

$$D_f = \mathbb{R} - \{x \in \mathbb{R} : x - 1 = 0\},$$

that is,

$$D_f =]-\infty, 1[\cup]1, +\infty[.$$

Exercise 1.5.1. *Determine the domain of definition of the following real-valued functions:* $f(x) = \sqrt{x^2 - x - 2}$; $g(x) = \dfrac{1}{\sqrt{x} + \sqrt{2-x}}$

Solution 1.5.1. $D_f =]-\infty, -1] \cup [2, +\infty[$ *and* $D_g = [0, 2]$.

Definition 1.5.2. *Let $f : E \longrightarrow F$ and $A \subset E$. The function g defined by:*

$$\begin{aligned} g : A &\longrightarrow F \\ x &\longmapsto g(x) = f(x) \end{aligned}$$

is called the restriction of f to A, sometimes denoted by $f|_A$. Conversely, f is called an extension of g to E.

Example 1.5.3. *The function*

$$g = \sin : [0, 2\pi] \longrightarrow \mathbb{R}$$
$$x \longmapsto \sin x$$

is the restriction of

$$f = \sin : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \sin x.$$

to the interval $[0, 2\pi]$.

Graph of a Real-Valued Function

The graph of a function $f : E \longrightarrow F$ is a subset of $\mathbb{R}^2$ defined by

$$G(f) = \{(x, f(x)) : x \in E\},$$

which is represented in the plane with respect to an orthonormal coordinate system $(O, \overrightarrow{i}, \overrightarrow{j})$.

1.6 Absolute Value of a Real Number

We denote by $\mathbb{R}_+$ the set of all non-negative real numbers:

$$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}.$$

We denote by $\mathbb{R}_-$ the set of all non-positive real numbers:

$$\mathbb{R}_- = \{x \in \mathbb{R} \mid x \leq 0\}.$$

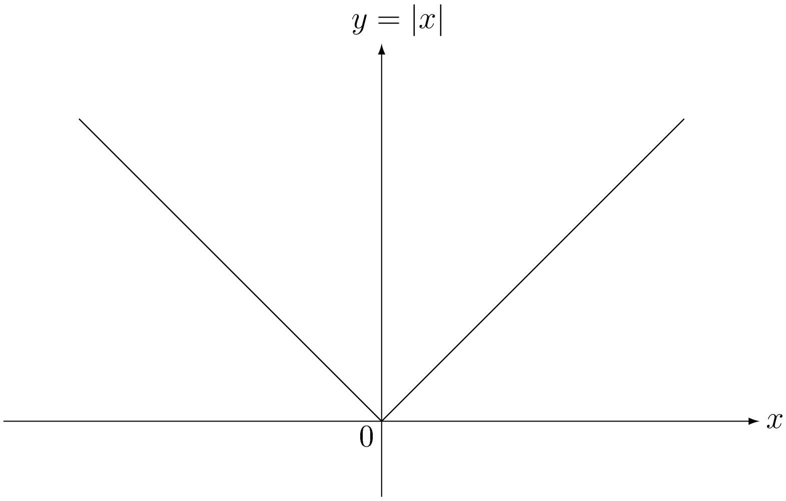
It follows that

$$\mathbb{R}_+ \cup \mathbb{R}_- = \mathbb{R} \quad \text{and} \quad \mathbb{R}_+ \cap \mathbb{R}_- = \{0\}.$$

Definition 1.6.1.
We call absolute value the function from $\mathbb{R}$ *to* $\mathbb{R}_+$*, denoted by* $|\cdot|$*, defined by:*

$$|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}_+$$
$$x \longmapsto |x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Graphical Representation of the Absolute Value Function



Example 1.6.1. $|3| = 3$; $|\frac{31}{2}| = \frac{31}{2}$; $|-7| = -(-7) = 7$;
$|-\sqrt{3}| = -(-\sqrt{3}) = \sqrt{3} = |\sqrt{3}|$.

Exercise 1.6.1. *Write the following expressions without using the symbol "$|\cdot|$", according to the values of x:*

$$A = |x+2| \ ; \qquad B = |x+2| + |1-x| + |x+3|.$$

Exercise 1.6.2. *Solve in $\mathbb{R}$:*

$$1)\ |x+5| = 12 \ ; \qquad 2)\ |3-x| = 5.$$

Theorem 1.6.1. *The absolute value on $\mathbb{R}$ has the following properties:*

1) $|x| \geq 0$.

2) $|x| = 0 \Longleftrightarrow x = 0$.

3) $|x| = |-x|$.

4) $|xy| = |x||y|$.

5) $|\frac{x}{y}| = \frac{|x|}{|y|}$ *if* $y \neq 0$.

6) $|x+y| \leq |x|+|y|$ *(called the first triangular inequality).*

7) $||x|-|y|| \leq |x-y|$ *(called the second triangular inequality).*

8) $-|x| \leq x \leq |x|$.

9) $|x| \leq \alpha \Longleftrightarrow -\alpha \leq x \leq \alpha$ *with* $\alpha \geq 0$.

Note that $|x+y| \neq |x|+|y|$ in general. For example: $|-8+1| = 7 \neq |-8|+|1| = 9$.

Example 1.6.2. *Let* α *be a strictly positive real number.*

1. *Show that* $|x| < \alpha \Longleftrightarrow -\alpha < x < \alpha$.
2. *Deduce the solutions of the inequality* $|x+2| < 1$.
3. *Solve in* $\mathbb{R}$ *the inequality* $1-x^2 \geq 0$.

Proof.

1. We have:

$$\begin{aligned} 0 \leq |x| < \alpha &\Longleftrightarrow |x|^2 < \alpha^2 \\ &\Longleftrightarrow x^2-\alpha^2 < 0 \\ &\Longleftrightarrow x \in]-\alpha,\alpha[\quad \text{(according to the sign of a quadratic polynomial)} \\ &\Longleftrightarrow -\alpha < x < \alpha. \end{aligned}$$

2. We have:

$$\begin{aligned} |x+2| < 1 &\Longleftrightarrow -1 < x+2 < 1 \\ &\Longleftrightarrow -1-2 < x+2-2 < 1-2 \quad \text{(by adding } -2\text{)} \\ &\Longleftrightarrow -3 < x < -1, \end{aligned}$$

hence the set of solutions is $S =]-3,-1[$.

3. Since $\sqrt{x^2} = |x|$, we have:

$$1-x^2 \geq 0 \Longleftrightarrow x^2 \leq 1 \Longleftrightarrow |x| \leq 1 \Longleftrightarrow -1 \leq x \leq 1,$$

hence the set of solutions is $S = [-1,1]$.

1.7 The Floor Function of a Real Number

Given a real number x, there exists the greatest integer, denoted by $E(x)$ or $[x]$, such that $E(x) \leq x$. It is called the *floor* (or *integer part*) of x. By definition, we have:

$$\forall x \in \mathbb{R}, \quad E(x) \leq x < E(x) + 1$$

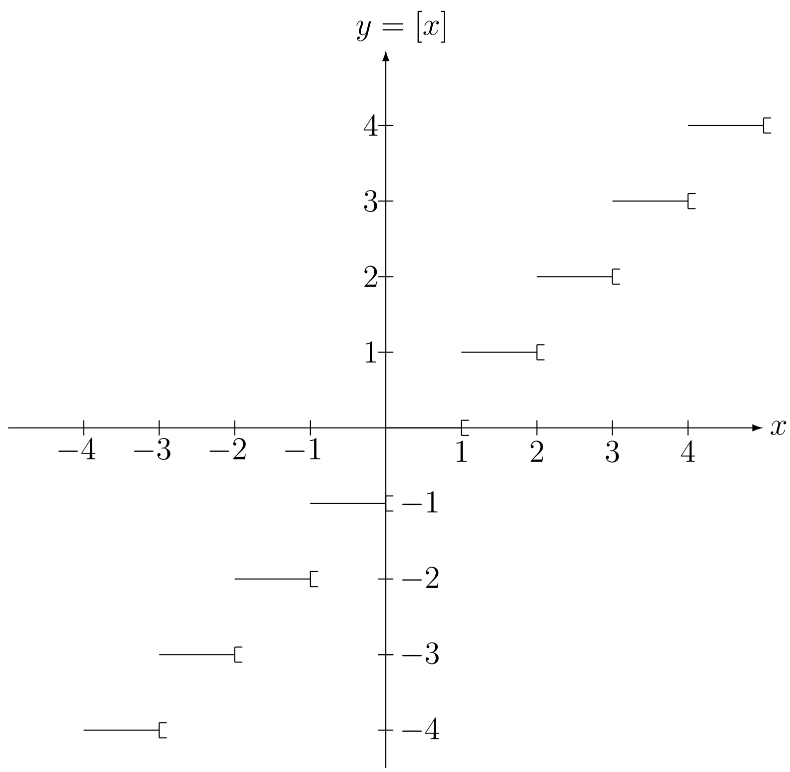
with $E(x) \in \mathbb{Z}$. Hence, we deduce that:

$$\forall x \in \mathbb{R}, \quad x - 1 < E(x) \leq x.$$

Remark 1.7.1. *If $x \in \mathbb{Z}$, then $E(x) = x$.*

Example 1.7.1. $[3,5] = 3$; $[0,7] = 0$; $[\frac{13}{7}] = 1$; $[-0,05] = -1$; $[-\sqrt{2}] = -2$; $E(-3,4) = E(-3,9) = E(-3,1) = E(-4) = -4$.

Graphical Representation of the Floor Function



1.8 Direct Image and Inverse Image of a Set

Definition 1.8.1. *Let* $f : E \longrightarrow F$, $A \subset E$ *and* $B \subset F$.

1. *The* ***direct image*** *of* A *under* f *is the subset of* F, *denoted by*

$$f(A) = \{\, f(x) \in F \mid x \in A \,\}.$$

2. *The* ***inverse image*** *of* B *under* f *is the subset of* E, *denoted by*

$$f^{-1}(B) = \{\, x \in E \mid f(x) \in B \,\}.$$

Example 1.8.1.

1. *Let the function*

$$\begin{aligned} f : \mathbb{R}_+^* &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \ln x. \end{aligned}$$

Find $f([1, +\infty))$, $f(\{1, 2\})$, $f^{-1}(\{1, 2\})$, $f^{-1}([1, +\infty))$.

2. *Let the function*

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = x^2. \end{aligned}$$

Find $f([-1, 1])$; $f((-\infty, 0))$; $f(\mathbb{R})$; $f^{-1}(\{1, 2, 4\})$; $f^{-1}((-3, -1))$; $f^{-1}((0, +\infty))$.

1.9 Composition of Functions

Let E, F, G be three subsets of $\mathbb{R}$, and let $f : E \longrightarrow F$ and $g : F \longrightarrow G$. The **composition of** f **with** g is the function from E to G, denoted by $g \circ f$, defined by

$$(g \circ f)(x) = g(f(x)), \quad \forall x \in E.$$

Exercise 1.9.1. *Let*

$$f(x) = x^2 \quad \text{and} \quad g(x) = \frac{1}{1+x}.$$

Compare $g \circ f$ *and* $f \circ g$ *on their respective domains of definition.*

Solution 1.9.1.

- *g is defined on $\mathbb{R}\setminus\{-1\}$ and f is defined on $\mathbb{R}$, so $f\circ g$ is defined on $\mathbb{R}\setminus\{-1\}$, and for all $x\in\mathbb{R}\setminus\{-1\}$,*
$$(f\circ g)(x) = f(g(x)) = f\left(\frac{1}{1+x}\right) = \frac{1}{(1+x)^2}.$$

- *f is defined on $\mathbb{R}$ and g is defined on $\mathbb{R}\setminus\{-1\}$. Hence, $g\circ f$ is defined for all values of x for which $f(x)\in\mathbb{R}\setminus\{-1\}$ (so that $g\circ f$ can be computed).*
Since $f(x)\geq 0$, we have $f(x)\in\mathbb{R}\setminus\{-1\}$. Therefore, $g\circ f$ is defined on $\mathbb{R}$, and for all $x\in\mathbb{R}$,
$$(g\circ f)(x) = g(f(x)) = g(x^2) = \frac{1}{1+x^2}.$$

We notice that, in general, $g\circ f\neq f\circ g$.

Definition 1.9.1. *Let f be a function defined on an interval I that is symmetric with respect to 0. We say that:*

1. *f is **even** if $\forall x\in I$: $f(-x)=f(x)$.*
2. *f is **odd** if $\forall x\in I$: $f(-x)=-f(x)$.*

Exercise 1.9.2. *Show that the function $x\longmapsto \exp x$ can be written as the sum of two functions g and h such that g is even and h is odd.*

Solution 1.9.2. *The function $x\longmapsto\exp x$ is defined on $\mathbb{R}$, which is symmetric with respect to 0. Moreover,*

$$\begin{cases} g(x)+h(x)=e^x \\ g(-x)+h(-x)=e^{-x}\end{cases} \Longrightarrow \begin{cases} g(x)+h(x)=e^x \\ g(x)-h(x)=e^{-x} \quad \textit{(since g is even and h is odd)}\end{cases}$$

Solving this system, we find:

$$g(x)=\frac{e^x+e^{-x}}{2} \quad \textit{and} \quad h(x)=\frac{e^x-e^{-x}}{2}.$$

Definition 1.9.2. *Let $f:\mathbb{R}\longrightarrow\mathbb{R}$. The function f is called **periodic** with period p if p is the smallest positive number such that*

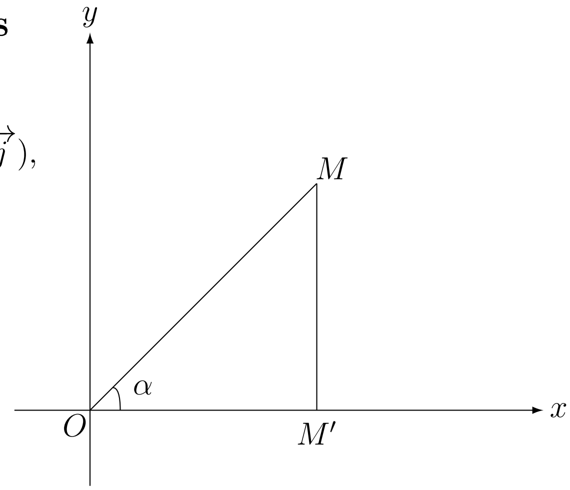
$$\forall x\in\mathbb{R}: \quad f(x+p)=f(x).$$

1.10 Trigonometric Functions

1.10.1 Sine and Cosine Functions

Introduction

In an orthonormal coordinate system $(O, \overrightarrow{i}, \overrightarrow{j})$,
consider the right triangle OMM'
such that $\widehat{MOM'} = \alpha$.
We have,
$\sin\alpha = \dfrac{MM'}{OM} = \dfrac{\text{opposite side}}{\text{hypotenuse}}$,
and
$\cos\alpha = \dfrac{OM'}{OM} = \dfrac{\text{adjacent side}}{\text{hypotenuse}}$.



According to the Pythagorean theorem,

$$OM'^2 + MM'^2 = OM^2$$

because OMM' is a right triangle.
Substituting, we get

$$OM^2\cos^2\alpha + OM^2\sin^2\alpha = OM^2,$$

which leads to

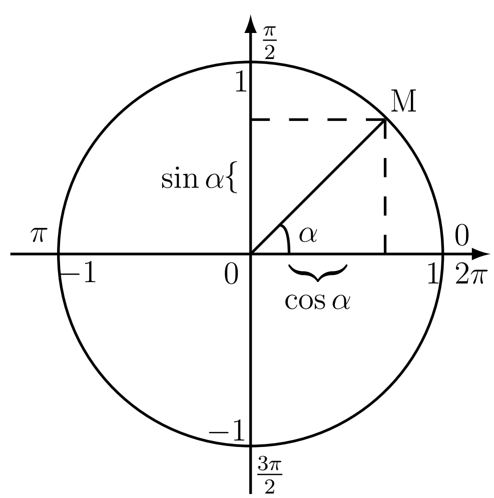
$$\cos^2\alpha + \sin^2\alpha = 1.$$

This is the equation of a circle with center 0 and radius 1.

Geometric Interpretation

The trigonometric circle has a radius of 1.
$\cos\alpha$ is the projection of the point M onto the x-axis.
$\sin\alpha$ is the projection of the point M onto the y-axis.

$$-1 \le \sin\alpha \le 1\ , \quad -1 \le \cos\alpha \le 1$$



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0

Table 1.1: trigonometric lines

We observe that for $\alpha \in [2\pi, 4\pi]$, we obtain the same values, that is, the functions $\sin x$ and $\cos x$ are periodic with a period of 2π, since

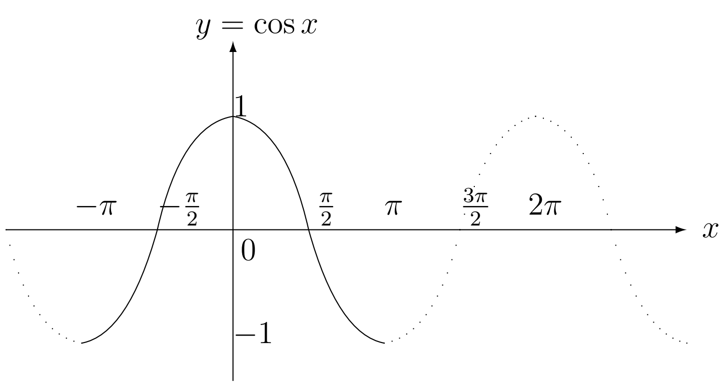
$$\forall x \in \mathbb{R} : \ f(x+2\pi) = \cos(x+2\pi) = \cos x = f(x) \ \text{ and } \ g(x+2\pi) = \sin(x+2\pi) = \sin x = g(x).$$

Therefore, we can restrict our study to an interval I of length 2π.

Graphical Interpretation

1) The variation table of $f(x) = \cos x$ with $f'(x) = -\sin x$

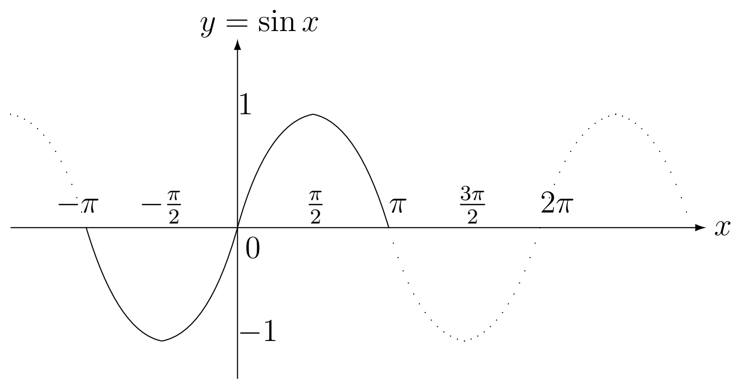
x	$-\pi$		0		π
$f'(x)$		$+$		$-$	
$f(x)$	-1	$\nearrow$	1	$\searrow$	-1



There is a symmetry with respect to the y-axis, i.e., $f(x) = \cos x$ is an even function and $\cos(-x) = \cos x$.

2) The variation table of $g(x) = \sin x$ with $g'(x) = \cos x$

x	$-\pi$		$-\frac{\pi}{2}$		$\frac{\pi}{2}$		π
$g'(x)$		$-$		$+$		$-$	
$g(x)$	0	$\searrow$	-1	$\nearrow$	1	$\searrow$	0



There is a symmetry with respect to the origin, i.e., $g(x) = \sin x$ is an odd function and $\sin(-x) = -\sin x$.

1.10.2 Tangent and Cotangent Functions

The tangent function, denoted by tan, is defined as follows:

$$\tan x = \frac{\sin x}{\cos x}, \ \forall x \in \mathbb{R} - A \quad \text{such that} \quad A = \{\frac{\pi}{2} + k\pi / k \in \mathbb{Z}\}.$$

The cotangent function, denoted by cot, is defined as follows:

$$\cot x = \frac{\cos x}{\sin x}, \ \forall x \in \mathbb{R} - B \quad \text{such that} \quad B = \{k\pi / k \in \mathbb{Z}\}.$$

For all $x \in \mathbb{R} \setminus (A \cup B)$, we have the following equality: $\cot x = \dfrac{1}{\tan x}$.

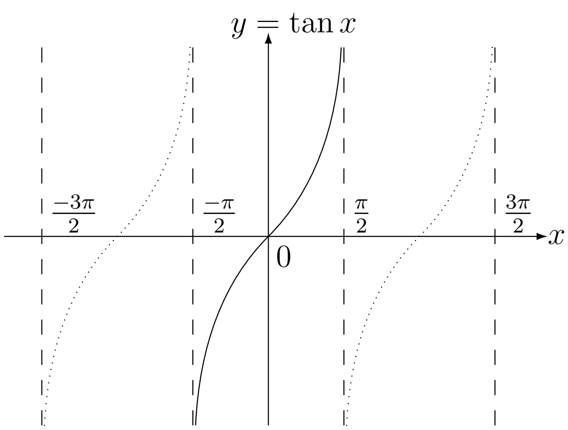
Graphical Interpretation

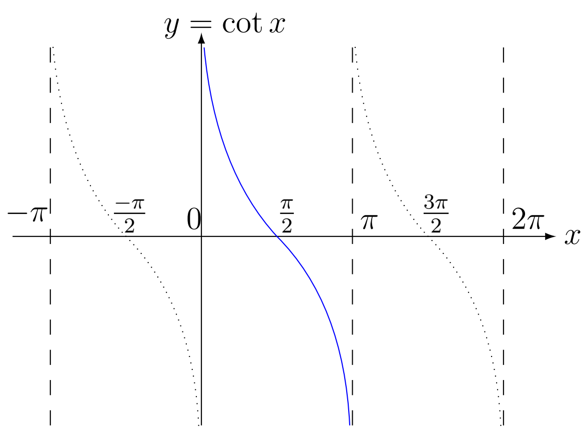
$$(\tan x)' = \frac{1}{\cos^2 x}\ ,\quad \forall x \in \mathbb{R} - A \qquad \text{and} \qquad (\cot x)' = \frac{-1}{\sin^2 x}\ ,\quad \forall x \in \mathbb{R} - B.$$

The functions $\tan x$ and $\cot x$ are periodic with a period of π. We can restrict our study to an interval I of length π. For example, we choose:
$J = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ for the function $\tan x$, and $J' =]0, \pi[$ for the function $\cot x$.

x	$-\frac{\pi}{2}$		0		$\frac{\pi}{2}$
$(\tan x)'$		$+$		$+$	
$\tan x$	$-\infty$	$\nearrow$	0	$\nearrow$	$+\infty$

x	0		$\frac{\pi}{2}$		π
$(\cot x)'$		$-$		$-$	
$\cot x$	$+\infty$	$\searrow$	0	$\searrow$	$-\infty$





There is a symmetry with respect to the origin, i.e., the function $\tan x$ is odd and $\tan(-x) = -\tan x$.
There is also a symmetry with respect to the origin, i.e., the function $\cot x$ is odd and $\cot(-x) = -\cot x$.

1.11 The Upper and Lower Bounds in $\mathbb{R}$

Let A be a non-empty subset of $\mathbb{R}$.

Definition 1.11.1. ***(of an upper bound)***
We say that the real number $M \in \mathbb{R}$ is an ***upper bound*** *of A if it satisfies: $\forall x \in A$, $x \leq M$.*
In this case, we say that A is ***bounded above****.*

Example 1.11.1. *Let $A = [0, 5[$. Then 6 is an upper bound since $\forall x \in A$, $x < 5 < 6$. We observe that every real number $M \geq 5$ is also an upper bound of A.*

Remark 1.11.1. *If M is an upper bound of A, then every $M' \geq M$ is also an upper bound of A.*

Definition 1.11.2. ***(of a lower bound)***
We say that the real number $m \in \mathbb{R}$ is a ***lower bound*** *of A if it satisfies: $\forall x \in A$, $x \geq m$.*
In this case, we say that A is ***bounded below****.*

Example 1.11.2. *Let $A = [0, 5[$. Then 0 is a lower bound of A, and we observe that every real number $m \leq 0$ is also a lower bound of A.*

Remark 1.11.2. *If m is a lower bound of A, then every $m' \leq m$ is also a lower bound of A.*

Definition 1.11.3. *We say that A is* ***bounded*** *if it is both bounded below and bounded above.*

Example 1.11.3. *The set $A =]-1, 3[\cup [4, 7[$ is bounded since $\forall x \in A: \ -1.5 \leq x \leq 7.2$. We observe that there are several upper and lower bounds.*

Definition 1.11.4. *A function $f : E \longrightarrow \mathbb{R}$ is said to be* ***bounded*** *if the set $f(E)$ is bounded. That is, there exist two real numbers M and m such that*

$$\forall x \in E, \ m \leq f(x) \leq M,$$

or equivalently, there exists $M > 0$ such that $\forall x \in E, \ |f(x)| \leq M$.

Definition 1.11.5. ***(of the least upper bound)***
*The **least upper bound** of A is the smallest of all upper bounds of A. It is denoted by $\sup A$. The $\sup A$, if it exists, is unique.*

Definition 1.11.6. ***(of the greatest lower bound)***
*The **greatest lower bound** of A is the largest of all lower bounds of A. It is denoted by $\inf A$. The $\inf A$, if it exists, is unique.*

Example 1.11.4.

1. $A =]-2, 3[$, $\inf(A) = -2$, $\sup(A) = 3$.

2. $B =]0, +\infty[$, $\inf(B) = 0$, $\sup(B) = +\infty$.

Remark 1.11.3. *If a subset B of $\mathbb{R}$ is not bounded above (or not bounded below), we set $\sup B = +\infty$ (or $\inf B = -\infty$).*

Remark 1.11.4. *If M is an upper bound of A and belongs to A, then it is the supremum of A, and in this case $\sup A = \max A = M$.*
If m is a lower bound of A and belongs to A, then it is the infimum of A, and in this case $\inf A = \min A = m$.
If $\sup A$ exists but does not belong to A, then $\max A$ does not exist.
If $\inf A$ exists but does not belong to A, then $\min A$ does not exist.

Exercise 1.11.1. *Let A be a subset of $\mathbb{R}$, defined as follows:*

$$A = \{x^2 e^{\frac{6}{x}} \ / \ 1 \leq x \leq 10^5\}.$$

1. *Compute $\sup A$, $\inf A$.*

2. *Do $\max A$ and $\min A$ exist?*

Solution 1.11.1.

1. *Let us find $\sup A$ and $\inf A$ using the variation table. Set $f(x) = x^2 e^{\frac{6}{x}}$, hence $f'(x) = e^{\frac{6}{x}}(2x - 6)$.*

x	1		3		10^5
$f'(x)$		$-$		$+$	
$f(x)$	e^6	$\searrow$	$9e^2$	$\nearrow$	$10^{10}e^{\frac{6}{10^5}}$

We observe from the table that $\inf A = 9e^2$ *and* $\sup A = 10^{10}e^{\frac{6}{10^5}}$ *(note that* $e^6 < 10^{10}e^{\frac{6}{10^5}}$*).*

2. $\max A = \sup A = 10^{10}e^{\frac{6}{10^5}}$ *since* $\sup A \in A$ *for* $x = 10^5$.
   $\min A = \inf A = 9e^2$ *since* $\inf A \in A$ *for* $x = 3$.

1.12 Binomial Theorem

Definition 1.12.1. *The factorial of a natural number is defined as:*
$n! = n(n-1)(n-2)\dots 2\cdot 1 = n(n-1)!$ *with* $1! = 1$ *and* $0! = 1$.
The integer $C_n^k = \dfrac{n!}{k!(n-k)!}$ *is the number of combinations of* k *elements taken from* n.
By definition, we have $C_n^0 = C_n^n = 1$.

Theorem 1.12.1. *(Binomial Theorem)*
For all real numbers x, y *and for every integer* $n \geq 1$, *we have the following binomial formula:*

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n} C_n^k x^{n-k} y^k \\ &= x^n + C_n^1 x^{n-1} y + \ldots + C_n^p x^{n-p} y^p + \ldots + y^n.\end{aligned}$$

Particular case: $(x+y)^2 = x^2 + C_2^1 xy + y^2 = x^2 + 2xy + y^2$.

Exercise 1.12.1. *Apply the Binomial Theorem to the expression* $(1+x)^n$.
Deduce that $C_n^0 + C_n^1 + \ldots + C_n^n = 2^n$.

Solution 1.12.1. $(x+1)^n = x^n + C_n^1 x^{n-1} + \ldots + C_n^{n-1} x + 1$.
Substituting $x = 1$, *we obtain:* $2^n = 1 + C_n^1 + \ldots + C_n^{n-1} + 1 = C_n^0 + C_n^1 + \ldots + C_n^{n-1} + C_n^n$.

Exercise 1.12.2. *Expand the following expressions:* $(x+2)^3$ *and* $(x-2)^4$.

Chapter 2

Limits and Continuity of Functions

2.1 Limits

Definition 2.1.1. *Let f be a function defined in a neighborhood of $x_0 \in \mathbb{R}$. We say that f has a (finite) limit l at the point x_0 if:*

$$\forall \epsilon > 0,\ \exists \alpha > 0,\ \forall x,\ [|x - x_0| < \alpha \Longrightarrow |f(x) - l| < \epsilon],$$

and we write:

$$\lim_{x \longrightarrow x_0} f(x) = l \quad \textit{or simply} \quad f \longrightarrow l \textit{ as } x \longrightarrow x_0.$$

Example 2.1.1. *Using the definition of the limit, show that:*

1) $\lim\limits_{x \longrightarrow 1} (2x - 3) = -1.$

2) $\lim\limits_{x \longrightarrow 0} (x^2) = 0.$

Proof.

1) Let $\epsilon > 0$. We have

$$|(2x - 3) + 1| = 2|x - 1| < \epsilon \Longleftrightarrow |x - 1| < \frac{\epsilon}{2},$$

hence

$$\forall \epsilon > 0,\ \exists \alpha = \frac{\epsilon}{2},\ \forall x \in \mathbb{R},\ \left[|x - 1| < \frac{\epsilon}{2} \Longrightarrow |f(x) - (-1)| < \epsilon\right].$$

2) Let $\epsilon > 0$. We have

$$|x| < \alpha \Longrightarrow |x^2| < \alpha^2 = \epsilon,$$

hence

$$\forall \epsilon > 0,\ \exists \alpha = \sqrt{\epsilon},\ \forall x \in \mathbb{R},\ \big[|x| < \sqrt{\epsilon} \Longrightarrow |x^2| < \epsilon\big].$$

2.1.1 Right-hand and Left-hand Limits

Let f be a function defined in a neighborhood of x_0. We say that:

1. f has a right-hand limit at x_0, denoted by $\lim_{x \longrightarrow x_0^+} f(x) = l$, if

$$\forall \epsilon > 0,\ \exists \eta > 0,\ \forall x,\ [x_0 < x < x_0 + \eta \Longrightarrow |f(x) - l| < \epsilon].$$

2. f has a left-hand limit at x_0, denoted by $\lim_{x \longrightarrow x_0^-} f(x) = l$, if

$$\forall \epsilon > 0,\ \exists \eta > 0,\ \forall x,\ [x_0 - \eta < x < x_0 \Longrightarrow |f(x) - l| < \epsilon].$$

3. The function f has a limit l at x_0 if and only if $\lim_{x \longrightarrow x_0^+} f(x) = \lim_{x \longrightarrow x_0^-} f(x) = l$.

Example 2.1.2. *Calculate the following limit:* $\lim_{x \longrightarrow 0} \dfrac{|x|}{x}$.

Proof.
We have: $\lim_{x \longrightarrow 0^+} \dfrac{|x|}{x} = \lim_{x \longrightarrow 0^+} \dfrac{x}{x} = 1$ and $\lim_{x \longrightarrow 0^-} \dfrac{|x|}{x} = \lim_{x \longrightarrow 0^-} \dfrac{-x}{x} = -1$.

Since $\lim_{x \longrightarrow 0^+} \dfrac{|x|}{x} \neq \lim_{x \longrightarrow 0^-} \dfrac{|x|}{x}$, the function f has no limit at 0.

2.2 General Properties of Function Limits

Theorem 2.2.1. *(Uniqueness of the Limit)*
If f admits a limit l at the point x_0, then this limit is unique.

Theorem 2.2.2. *Let f and g be two functions defined from E to $\mathbb{R}$ such that $\lim\limits_{x \longrightarrow x_0} f(x) = l$ and $\lim\limits_{x \longrightarrow x_0} g(x) = l'$ (or $x_0 = \pm\infty$). Then:*

1. $\lim\limits_{x \longrightarrow x_0} [f(x) + g(x)] = l + l'$ *(except in the case $l = +\infty$ and $l' = -\infty$).*

2. $\lim\limits_{x \longrightarrow x_0} (\lambda f(x)) = \lambda l$, $\forall \lambda \in \mathbb{R}$.

3. $\lim\limits_{x \longrightarrow x_0} [f(x)g(x)] = ll'$ *(except in the case $l = \pm\infty$ and $l' = 0$).*

4. $\lim\limits_{x \longrightarrow x_0} |f(x)| = |l|$.

5. $\lim\limits_{x \longrightarrow x_0} \dfrac{f(x)}{g(x)} = \dfrac{l}{l'}$ *where $l' \neq 0$ and $g(x) \neq 0$, for x in a neighborhood of x_0.*

Theorem 2.2.3. *Let $f : E \to F$ and $g : F \to G$.*
If $\lim\limits_{x \longrightarrow x_0} f(x) = y_0$ and $\lim\limits_{y \longrightarrow y_0} g(y) = l$, then $\lim\limits_{x \longrightarrow x_0} (g \circ f)(x) = \lim\limits_{x \longrightarrow x_0} g(f(x)) = l$.

Example 2.2.1. $(g \circ f)(x) = e^{\cos x} \underset{x \longrightarrow 0}{\longrightarrow} e$ *since* $f(x) = \cos x \underset{x \longrightarrow 0}{\longrightarrow} 1$ *and* $g(y) = e^y \underset{y \longrightarrow 1}{\longrightarrow} e$.

Theorem 2.2.4 (Squeeze Theorem or Sandwich Theorem)**.**

1. *Let f, g, and h be three functions defined in a neighborhood of x_0 such that $g \leq f \leq h$ in a neighborhood of x_0. If g and h have the same limit l as $x \longrightarrow x_0$, then $\lim\limits_{x \longrightarrow x_0} f(x) = l$.*

2. *Let f and g be two functions defined in a neighborhood of x_0 such that $f \leq g$ in a neighborhood of x_0.*

   - *If $\lim\limits_{x \longrightarrow x_0} f(x) = +\infty$, then $\lim\limits_{x \longrightarrow x_0} g(x) = +\infty$.*
   - *If $\lim\limits_{x \longrightarrow x_0} g(x) = -\infty$, then $\lim\limits_{x \longrightarrow x_0} f(x) = -\infty$.*

Example 2.2.2. *Compute the following limit:*

$$\lim_{x \to 0} x \cos\left(\frac{1}{x}\right).$$

Proof. We know that $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$. For $x > 0$, this implies

$$-x \leq x \cos\left(\frac{1}{x}\right) \leq x.$$

Hence,

$$\lim_{x\to 0^+}(-x) \leq \lim_{x\to 0^+} x\cos\left(\frac{1}{x}\right) \leq \lim_{x\to 0^+} x,$$

which gives $\lim_{x\to 0^+} x\cos\left(\frac{1}{x}\right) = 0.$
Similarly, for $x < 0$ we obtain

$$\lim_{x\to 0^-} x\cos\left(\frac{1}{x}\right) = 0.$$

Therefore,

$$\boxed{\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0.}$$

Exercise 2.2.1. *Show that* $\lim_{x\longrightarrow 0}\frac{\sin x}{x} = 1.$

Proof.
We have

$$\forall x \in \left[0, \frac{\pi}{2}\right[\ ,\ \sin x \leq x \leq \tan x,$$

because, setting $h(x) = \sin x - x$, we have $h'(x) = \cos x - 1 \leq 0$,

x	0	$\frac{\pi}{2}$
$h'(x)$	$-$	
$h(x)$	0 $\searrow$	$1-\frac{\pi}{2}$

thus $h(x) \leq 0$, and for $g(x) = \tan x - x$, we have $g'(x) = \frac{1-\cos^2 x}{\cos^2 x} \geq 0$ since $\cos^2 x \leq 1$.

x	0	$\frac{\pi}{2}$
$g'(x)$	$+$	
$g(x)$	0 $\nearrow$	$+\infty$

Thus, $\sin x \leq x \leq \tan x$, and by dividing by $\sin x$ for $x > 0$, we obtain

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}.$$

By taking reciprocals, we get
$$\cos x \leq \frac{\sin x}{x} \leq 1.$$
Since $\lim\limits_{x\to 0^+} \cos x = 1$, it follows from the Squeeze Theorem that
$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$
It remains to compute $\lim\limits_{x\to 0^-} \dfrac{\sin x}{x}$. If we let $x = -u$, then
$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{u\to 0^+} \frac{\sin(-u)}{-u} = \lim_{u\to 0^+} \frac{\sin u}{u} = 1 \quad \text{(since sin is an odd function).}$$
Finally,
$$\lim_{x\to 0^+} \frac{\sin x}{x} = \lim_{x\to 0^-} \frac{\sin x}{x} = 1,$$
hence
$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$
Proposition 2.2.1. *Let f and g be two functions defined in a neighborhood of x_0. If g is bounded in a neighborhood of x_0 and $\lim\limits_{x\longrightarrow x_0} f(x) = 0$, then*
$$\lim_{x\longrightarrow x_0} f(x)g(x) = 0.$$
Example 2.2.3. *Compute the following limit:*
$$\lim_{x\longrightarrow 0} x \sin\left(\frac{1}{x}\right).$$
Proof.
Observe that $\sin\left(\dfrac{1}{x}\right)$ is bounded since $-1 \leq \sin\left(\dfrac{1}{x}\right) \leq 1$ for $x \neq 0$. As $\lim\limits_{x\longrightarrow 0} x = 0$, we obtain
$$\lim_{x\longrightarrow 0} x \sin\left(\frac{1}{x}\right) = 0.$$

2.3 Indeterminate Forms

Let $x_0 \in \mathbb{R}$ or $x_0 = \infty$. The following cases are called *indeterminate forms* because the resulting limit may take any possible value.

1) If $\lim\limits_{x \longrightarrow x_0} f(x) = +\infty$ and $\lim\limits_{x \longrightarrow x_0} g(x) = +\infty$, then $f - g$ and $\dfrac{f}{g}$ may have any possible behavior near x_0.

2) If $\lim\limits_{x \longrightarrow x_0} f(x) = +\infty$ and $\lim\limits_{x \longrightarrow x_0} g(x) = 0$, then $f \cdot g$ and f^g may have any possible behavior near x_0.

3) If $\lim\limits_{x \longrightarrow x_0} f(x) = 0$ and $\lim\limits_{x \longrightarrow x_0} g(x) = 0$, then $\dfrac{f}{g}$ and f^g may have any possible behavior near x_0.

4) If $\lim\limits_{x \longrightarrow x_0} f(x) = 1$ and $\lim\limits_{x \longrightarrow x_0} g(x) = \infty$, then f^g may have any possible behavior near x_0.

Exercise 2.3.1. *Compute the following limits:*

1) $\lim\limits_{x \longrightarrow 0} \dfrac{x^2 \sin(\frac{1}{x})}{\sin x}$ 2) $\lim\limits_{x \longrightarrow a} \dfrac{\sqrt{x} - \sqrt{a} - \sqrt{x-a}}{\sqrt{x^2 - a^2}}$ 3) $\lim\limits_{x \longrightarrow +\infty} xE(\frac{1}{x})$

4) $\lim\limits_{x \longrightarrow 0} \dfrac{\sin(2x)}{x}$ 5) $\lim\limits_{x \longrightarrow +\infty} (2\sqrt{x} + \sqrt{x} \sin x)$ 6) $\lim\limits_{x \longrightarrow +\infty} (1 + \dfrac{3}{x})^x$

7) $\lim\limits_{x \longrightarrow +\infty} x^2 \cos(\frac{1}{x})$ 8) $\lim\limits_{x \longrightarrow +\infty} (x - \sqrt{1 + x^2})$ 9) $\lim\limits_{x \longrightarrow 0} \dfrac{\sin(px)}{\sin(qx)}$ *such that* $p, q \in \mathbb{R}^*$.

2.4 Continuous Functions

Definition 2.4.1. *Let f be a function from E to $\mathbb{R}$ and let $x_0 \in E$. We say that f is continuous:*

1. *at the point x_0 if $\lim\limits_{x \longrightarrow x_0} f(x) = f(x_0)$.*
2. *from the right at x_0 if $\lim\limits_{x \longrightarrow x_0^+} f(x) = f(x_0)$.*

3. *from the left at* x_0 *if* $\lim\limits_{x \longrightarrow x_0^-} f(x) = f(x_0)$.

It is clear that continuity from the right and from the left at x_0 *implies the continuity of* f *at* x_0.

Definition 2.4.2. *A function* f *defined on an interval* I *is said to be continuous on* I *if it is continuous at every point of* I.
The set of continuous functions on I *is denoted by* $C(I)$.

Remark 2.4.1. *Continuity is studied only in the case where* $x_0 \in D_f$.
A function that is continuous on an interval has a graphical representation that can be drawn with an unbroken line (without lifting the pencil from the paper) from the lower bound to the upper bound of the interval.

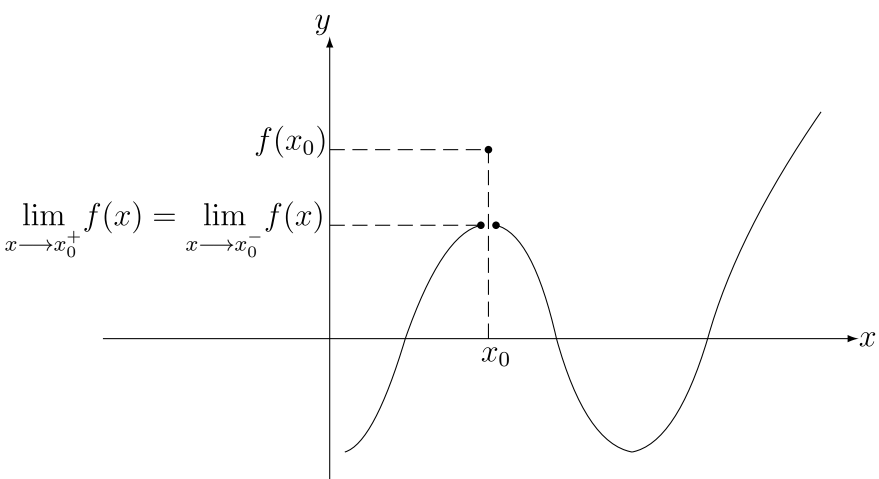
Example 2.4.1.

1. $f(x) = c$ $(c \in \mathbb{R})$, f *is continuous on* $\mathbb{R}$.

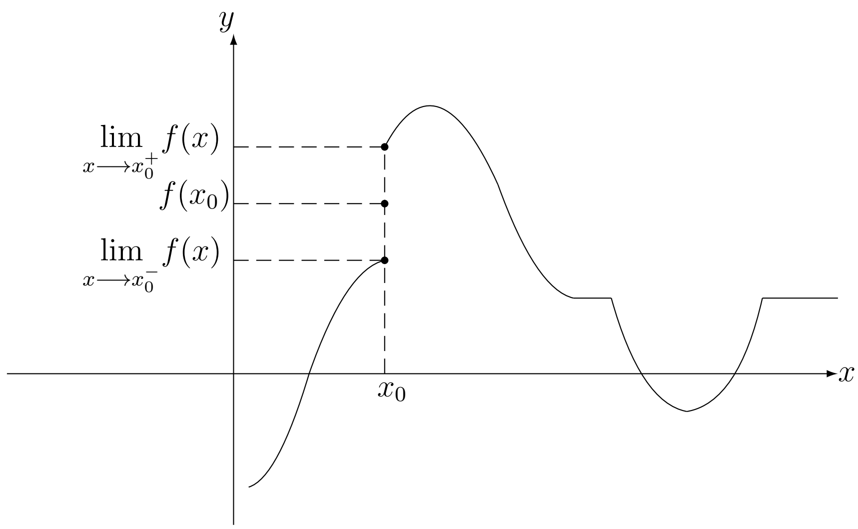
2. $f(x) = x$, f *is continuous on* $\mathbb{R}$.

3. $f(x) = \sin x$, f *is continuous on* $\mathbb{R}$.

Remark 2.4.2. *If* $\lim\limits_{x \longrightarrow x_0^+} f(x) \neq f(x_0)$, *then* f *is not continuous at* x_0.



If $\lim\limits_{x \longrightarrow x_0^+} f(x) \neq \lim\limits_{x \longrightarrow x_0^-} f(x) \Longrightarrow \lim\limits_{x \longrightarrow x_0} f(x) \nexists$. *Hence,* f *is not continuous at* x_0.



Theorem 2.4.1. *Let f and g be two functions defined on $I \subset \mathbb{R}$ and continuous at x_0 (respectively on I), then*

1. *$\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is continuous at x_0 (respectively on I).*
2. *If $g(x) \neq 0, \forall x \in I$ then $\dfrac{f}{g}$ is continuous at x_0 (respectively on I).*
3. *fg and $|f|$ are continuous at x_0 (respectively on I).*

Remark 2.4.3.
The function $f(x) = x$ is continuous on $\mathbb{R}$, therefore $g(x) = x^n$ is continuous on $\mathbb{R}$, $\forall n \in \mathbb{N}^*$. Thus $h(x) = \sum_{i=1}^{n} a_i x^i$ is a continuous polynomial on $\mathbb{R}$.

Exercise 2.4.1.
Study the continuity of the following functions on their domains of definition:

1) $f(x) = \begin{cases} 1 & , x > 0 \\ 0 & , x \leq 0 \end{cases}$; *2)* $g(x) = \begin{cases} x^2 + x & , x \leq 0 \\ \sin x & , 0 < x \leq \pi \\ 1 + \cos x & x > \pi \end{cases}$; *3)* $h(x) = [x]$

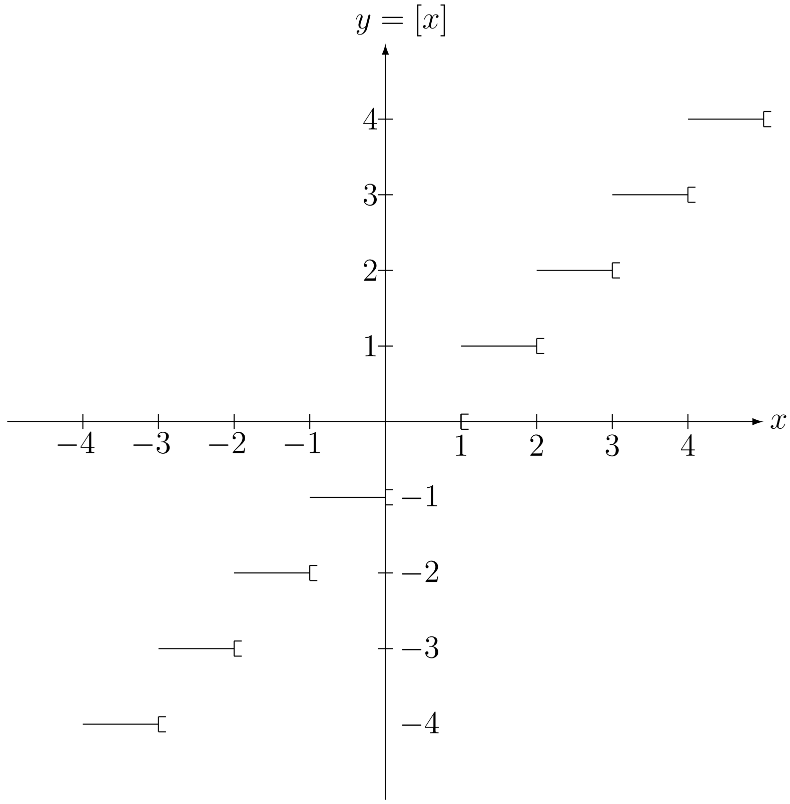
4) $k(x) = \begin{cases} x - a & , x \geq 0 \\ \dfrac{e^x - 1}{\sin x} & , -\pi < x < 0 \end{cases}$

Proof.

1. The function $x \longmapsto 1$ is continuous on $\mathbb{R}_+^*$ since it is constant, and the function $x \longmapsto 0$ is also continuous on $\mathbb{R}_-^*$ since it is constant. Therefore, the function f is continuous on $\mathbb{R}^*$. It remains to study the continuity at the point 0. We have $\lim\limits_{x \longrightarrow 0^+} f(x) = 1 \neq f(0) = 0$. Hence, f is not continuous at 0. That is, f is continuous on $\mathbb{R}^*$.

2. The function $x \longmapsto x^2 + x$ is continuous on $\mathbb{R}_-^*$ since it is a polynomial, and the function $x \longmapsto \sin x$ is continuous on $]0, \pi[$ since sin is continuous on $\mathbb{R}$, and the function $x \longmapsto 1+\cos x$ is continuous on $]\pi, +\infty[$ since it is the sum of two continuous functions ($x \longmapsto 1$, $x \longmapsto \cos x$) on $\mathbb{R}$. It remains to study the continuity at the points 0 and π. We have $g(0) = 0$ and $g(\pi) = 0$, and
$$\lim_{x \longrightarrow 0^-} g(x) = \lim_{x \longrightarrow 0^-} x^2 + x = 0 = g(0) = \lim_{x \longrightarrow 0^+} g(x) = \lim_{x \longrightarrow 0^+} \sin x.$$
Hence, g is continuous at 0.
$$\lim_{x \longrightarrow \pi^-} g(x) = \lim_{x \longrightarrow \pi^-} \sin x = 0 = g(\pi) = \lim_{x \longrightarrow \pi^+} g(x) = \lim_{x \longrightarrow \pi^+} (1 + \cos x).$$
Therefore, g is continuous at π. Consequently, g is continuous on $\mathbb{R}$.

3. The function $x \longmapsto [x]$ is continuous on $\mathbb{R} \setminus \mathbb{Z}$, i.e., for $x_0 \in \mathbb{R} \setminus \mathbb{Z}$, there exists $n = h(x_0) = [x_0]$ such that $x_0 \in]n, n+1[$ and
$$\lim_{x \longrightarrow x_0^+} [x] = n = [x_0] = h(x_0),$$
so h is continuous from the right at x_0.
$$\lim_{x \longrightarrow x_0^-} [x] = n = [x_0] = h(x_0),$$
so h is continuous from the left at x_0.
Now, let us study the continuity on $\mathbb{Z}$. Let $x_0 \in \mathbb{Z}$, then $h(x_0) = [x_0] = x_0$ and
$$\lim_{x \longrightarrow x_0^+} [x] = x_0 = h(x_0) \neq \lim_{x \longrightarrow x_0^-} [x] = x_0 - 1.$$
Hence, h is not continuous on $\mathbb{Z}$. Therefore, h is continuous on $\mathbb{R} \setminus \mathbb{Z}$.



Theorem 2.4.2 (Composition of continuous functions).
Let E and F be two subsets of $\mathbb{R}$, $f : E \longrightarrow F$ and $g : F \longrightarrow \mathbb{R}$ such that $f(E) \subseteq F$. Let $a \in E$. If f is continuous at a and g is continuous at $f(a)$, then the function $g \circ f$ is continuous at a.

Example 2.4.2. *Study the continuity of the function $x \longmapsto h(x) = \ln(x - x^2)$.*

Solution 2.4.1. *We have $h = g \circ f$, which is a composition of two continuous functions $x \longmapsto f(x) = x - x^2$ on $]0, 1[$ and $x \longmapsto g(x) = \ln x$ on $f(]0, 1[) =]0, \frac{1}{4}] \subset]0, +\infty[$. Therefore, h is continuous on $]0, 1[$.*

Remark 2.4.4.
If f is not continuous at a or g is not continuous at $f(a)$, this does not necessarily imply that $g \circ f$ is not continuous at a.

Example 2.4.3. *Let f and g be two functions defined by*

$$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases} \quad \textit{and} \quad g(x) = x^2, \quad \forall x \in \mathbb{R}.$$

We have f is not continuous at 0 and g is continuous at $f(0) = 1$, but the function

$$x \longmapsto (g \circ f)(x) = 1, \quad \forall x \in \mathbb{R},$$

is continuous at 0.

Example 2.4.4. *Let f and g be two functions defined by*

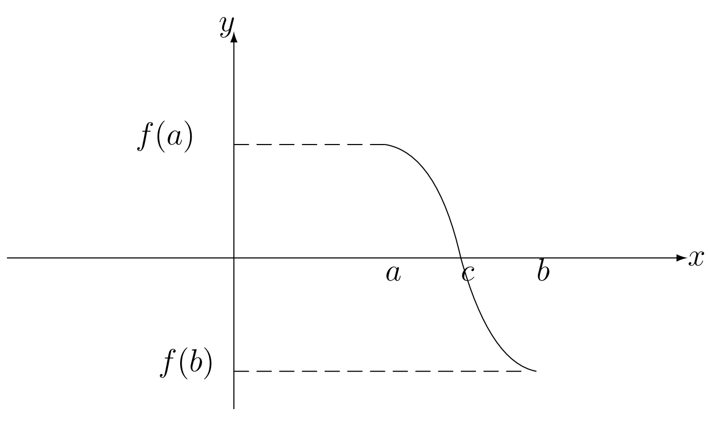
$$f(x) = 0 \quad \textit{and} \quad g(x) = \begin{cases} 1 - x, & x \geq 0 \\ -1, & x < 0 \end{cases}, \quad \forall x \in \mathbb{R}.$$

We have f is continuous at 0 and g is not continuous at $f(0) = 0$, but the function

$$x \longmapsto (g \circ f)(x) = 1, \quad \forall x \in \mathbb{R},$$

is continuous at 0.

Theorem 2.4.3 (Intermediate Value Theorem)**.**
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. If $f(a)$ and $f(b)$ have opposite signs, then there exists at least one point $c \in]a, b[$ such that $f(c) = 0$.



Example 2.4.5. *Show that the equation $x - 2 + \ln x = 0$ has a solution on the interval $]1, \sqrt{e}[$, and then prove that this solution is unique (note: $\sqrt{e} \approx 1.64$).*

Solution 2.4.2. *Consider the function* $f(x) = x - 2 + \ln x$.
Since f *is the sum of continuous functions, it is continuous on* $[1, \sqrt{e}]$.
We have

$$f(1) = -1 < 0 \quad \textit{and} \quad f(\sqrt{e}) \approx 0.14 > 0.$$

By the Intermediate Value Theorem, there exists at least one point $c \in]1, \sqrt{e}[$ *such that* $f(c) = 0$.
To prove uniqueness, we examine the derivative:

$$f'(x) = 1 + \frac{1}{x} > 0 \quad \textit{for all } x > 0,$$

so f *is strictly increasing. Therefore, the solution* c *is unique.*

Remark 2.4.5.

1. *If* f *is continuous on the closed interval* $[a, b]$, *then the image of* f *is the interval*

$$f([a, b]) = [m, M], \quad \textit{where} \quad m = \inf_{x \in [a,b]} f(x), \quad M = \sup_{x \in [a,b]} f(x).$$

2. *If* f *is continuous and increasing on* $[a, b]$, *then*

$$f([a, b]) = [f(a), f(b)].$$

3. *If* f *is continuous and decreasing on* $[a, b]$, *then*

$$f([a, b]) = [f(b), f(a)].$$

In general, one has

$$f([a, b]) \neq [f(a), f(b)].$$

Example 2.4.6. $f(x) = x^2$, $f([-1, 1]) = [0, 1] \neq [f(-1), f(1)] = \{1\}$.

2.5 Extension by Continuity

Definition 2.5.1. *Let $f : I \setminus \{x_0\} \longrightarrow \mathbb{R}$ be a function and suppose that*

$$\lim_{x \to x_0} f(x) = l.$$

Then the function

$$\tilde{f} : I \longrightarrow \mathbb{R}, \quad \tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in I \setminus \{x_0\}, \\ l, & \text{if } x = x_0, \end{cases}$$

is continuous at x_0. The function $\tilde{f}$ is called the extension of f by continuity *at the point x_0.*

Example 2.5.1.

1. $f(x) = \dfrac{\sin x}{x}$ *is defined and continuous on $D_f = \mathbb{R}^*$ as the quotient of two continuous functions on $\mathbb{R}^*$. We know that*

$$\lim_{x \longrightarrow 0} \frac{\sin x}{x} = 1,$$

   so f admits a continuous extension at 0, given by

$$\tilde{f}(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$$

   which is continuous on $\mathbb{R}$.

2. $g(x) = \dfrac{x^2 - 9}{x - 3}$ *is defined and continuous on $D_g = \mathbb{R} \setminus \{3\}$ as the quotient of two continuous functions on $\mathbb{R} \setminus \{3\}$. We have*

$$\lim_{x \longrightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \longrightarrow 3} \frac{(x-3)(x+3)}{x - 3} = \lim_{x \longrightarrow 3} (x + 3) = 6,$$

   so g admits a continuous extension at 3, given by

$$\tilde{g}(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & \text{if } x \neq 3, \\ 6, & \text{if } x = 3, \end{cases}$$

   which is continuous on $\mathbb{R}$.

3. $h(x) = e^{\frac{x-1}{x^2}}$ *is defined and continuous on* $D_h = \mathbb{R}^*$ *as a composition and quotient of continuous functions on* $\mathbb{R}^*$. *We know that*

$$\lim_{x \longrightarrow 0} e^{\frac{x-1}{x^2}} = e^{-\infty} = 0,$$

so h *admits a continuous extension at* 0, *given by*

$$\tilde{h}(x) = \begin{cases} e^{\frac{x-1}{x^2}}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

which is continuous on $\mathbb{R}$.

2.6 Inverse Function

2.6.1 Injective, Surjective, and Bijective Functions

Definition 2.6.1. *Let* $f : E \longrightarrow F$. *We say that* f *is:*

1. ***injective*** *if* $\forall (x_1, x_2) \in E^2,\ [x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)]$;
2. ***surjective*** *if* $\forall y \in F,\ \exists x \in E : y = f(x)$;
3. ***bijective*** *if* f *is both injective and surjective. That is, the equation* $y = f(x)$ *has a* unique *solution for every* $y \in F$.

Definition 2.6.2. *Let*

$$f : E \longrightarrow F, \quad x \longmapsto f(x) = y$$

be a bijective function. The inverse function *of* f *is defined by*

$$f^{-1} : F \longrightarrow E, \quad y \longmapsto f^{-1}(y) = x,$$

such that

$$\forall x \in E,\ \forall y \in F, \quad f(x) = y \Longleftrightarrow x = f^{-1}(y),$$

and we have

$$\forall x \in E,\ (f^{-1} \circ f)(x) = x, \quad \forall y \in F,\ (f \circ f^{-1})(y) = y.$$

Example 2.6.1. *Consider the function*

$$f : \mathbb{R} \longrightarrow]1, +\infty[$$
$$x \longmapsto f(x) = e^x + 1.$$

1. *Show that f is bijective.*

2. *Give explicitly the inverse function f^{-1}.*

3. *Verify that $f^{-1} \circ f = I_{\mathbb{R}}$ and $f \circ f^{-1} = I_{]1,+\infty[}$.*

Proof.

1. We will show that f is both injective and surjective.
   Injectivity: Let $x_1, x_2 \in \mathbb{R}$. Suppose

   $$f(x_1) = f(x_2) \Longrightarrow e^{x_1} + 1 = e^{x_2} + 1 \Longrightarrow e^{x_1} = e^{x_2} \Longrightarrow x_1 = x_2,$$

   using the properties of the exponential function. Thus, f is injective.

   Surjectivity: Let $y \in]1, +\infty[$. We need x such that $f(x) = y$:

   $$f(x) = y \Longleftrightarrow e^x + 1 = y \Longleftrightarrow e^x = y - 1 > 0 \Longleftrightarrow x = \ln(y - 1).$$

   Hence, f is surjective. Since f is both injective and surjective, it is bijective.

2. The equation $y = f(x)$ gives the unique solution $x = \ln(y - 1)$ for any $y \in]1, +\infty[$. Therefore, the inverse function is

   $$f^{-1} :]1, +\infty[\longrightarrow \mathbb{R}$$
   $$y \longmapsto f^{-1}(y) = \ln(y - 1).$$

3. We have:

   $$f^{-1} \circ f : \mathbb{R} \longrightarrow \mathbb{R}, \quad (f^{-1} \circ f)(x) = f^{-1}[f(x)] = \ln(e^x + 1 - 1) = x = I_{\mathbb{R}}(x),$$

   and also

   $$f \circ f^{-1} :]1, +\infty[\longrightarrow]1, +\infty[,$$
   $$(f \circ f^{-1})(y) = f[f^{-1}(y)] = e^{\ln(y-1)} + 1 = y - 1 + 1 = y = I_{]1,+\infty[}(y).$$

Definition 2.6.3. *Let $f : X \longrightarrow \mathbb{R}$ and let $I \subset X$ be an interval.*

1. *The function f is said to be **increasing** (respectively **strictly increasing**) on I if*

$$\forall x_1, x_2 \in I,\ x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2),$$

i.e.

$$\forall x_1 \neq x_2,\ \frac{f(x_1) - f(x_2)}{x_1 - x_2} \geq 0$$

(respectively, $x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$, i.e. $\forall x_1 \neq x_2,\ \dfrac{f(x_1) - f(x_2)}{x_1 - x_2} > 0$).

2. *The function f is said to be **decreasing** (respectively **strictly decreasing**) on I if*

$$\forall x_1, x_2 \in I,\ x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2),$$

i.e.

$$\forall x_1 \neq x_2,\ \frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq 0$$

(respectively, $x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)$, i.e. $\forall x_1 \neq x_2,\ \dfrac{f(x_1) - f(x_2)}{x_1 - x_2} < 0$).

3. *If f is increasing or decreasing, we say that f is **monotone** on I.*

4. *If f is strictly increasing or strictly decreasing, we say that f is **strictly monotone** on I.*

Property 2.6.1. *If f is strictly monotone on I, then $f : I \longrightarrow \mathbb{R}$ is injective.*

Proof. Suppose, for instance, that f is strictly increasing on I. For all $x_1, x_2 \in I$, with $x_1 \neq x_2$, we have

$$\begin{cases} x_1 < x_2 \Longrightarrow f(x_1) < f(x_2), \\ x_1 > x_2 \Longrightarrow f(x_1) > f(x_2), \end{cases}$$

which implies $f(x_1) \neq f(x_2)$. Hence, f is injective.

Theorem 2.6.1 (Inverse Function Theorem).
Let I be an interval and let $f : I \longrightarrow \mathbb{R}$ be a continuous and strictly monotone function. Then f is a bijection from I onto $f(I)$, and its inverse function

$$f^{-1} : f(I) \longrightarrow I$$

is continuous and strictly monotone (with the same sense of monotonicity as f).

Remark 2.6.1. *In an orthonormal coordinate system, the graphs of f and f^{-1} are symmetric with respect to the line $y = x$.*

Example 2.6.2. *Find the inverse function of*

$$f :]-1,1[\longrightarrow \mathbb{R}, \quad f(x) = \frac{x}{1-x^2}.$$

The function f is defined, continuous, and strictly increasing on $]-1,1[$ (according to its variation table). Hence, it defines a bijection from $]-1,1[$ onto $f(]-1,1[) = \mathbb{R}$.

Let us find f^{-1}. For $y \in \mathbb{R}$, there exists a unique $x \in]-1,1[$ such that $y = f(x)$. We have:

$$\begin{aligned} f(x) = y &\Longleftrightarrow y = \frac{x}{1-x^2} \\ &\Longleftrightarrow yx^2 + x - y = 0. \end{aligned}$$

If $y = 0$, then $x = 0$, i.e. $f^{-1}(0) = 0$.
If $y \neq 0$, we obtain two possible solutions:

$$x = \frac{-1-\sqrt{1+4y^2}}{2y} \quad \text{and} \quad x' = \frac{-1+\sqrt{1+4y^2}}{2y}.$$

Since we are looking for $x \in]-1,1[$, the correct solution is

$$x = \frac{-1+\sqrt{1+4y^2}}{2y}.$$

We reject

$$x = \frac{-1-\sqrt{1+4y^2}}{2y},$$

because for instance, when $y = \frac{1}{2}$, we obtain $x = -1-\sqrt{2} \notin]-1,1[$.
The expression can also be written as

$$x = \frac{2y}{1+\sqrt{1+4y^2}}.$$

Therefore, the inverse function is defined by

$$\begin{aligned} f^{-1} : \mathbb{R} &\longrightarrow]-1,1[\\ x &\longmapsto f^{-1}(x) = \frac{2x}{1+\sqrt{1+4x^2}}. \end{aligned}$$

2.7 Equivalent Functions

Definition 2.7.1. *Let f and g be two functions defined in a neighborhood of x_0. We say that f* ***is negligible compared to*** *g* ***as*** *$x \longrightarrow x_0$, and we write*

$$f = o(g),$$

if

$$\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = 0,$$

where g does not vanish in a neighborhood of x_0.

Example 2.7.1.

1. $x^3 = o(x^2)$ *as* $x \longrightarrow 0$, *since* $\displaystyle\lim_{x \longrightarrow 0} \frac{x^3}{x^2} = 0$.
2. $\sin(x-1) - x + 1 = o(x-1)$ *as* $x \longrightarrow 1$, *since* $\displaystyle\lim_{x \longrightarrow 1} \frac{\sin(x-1) - x + 1}{x-1} = 0$.
3. $\displaystyle\frac{1}{x} = o\left(\frac{1}{x^2}\right)$ *as* $x \longrightarrow 0$, *since* $\displaystyle\lim_{x \longrightarrow 0} \frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x \to 0} \frac{x^2}{x} = 0$.
4. $\displaystyle\frac{1}{x^2} = o\left(\frac{1}{x}\right)$ *as* $x \longrightarrow +\infty$, *since* $\displaystyle\lim_{x \longrightarrow +\infty} \frac{\frac{1}{x^2}}{\frac{1}{x}} = \lim_{x \longrightarrow +\infty} \frac{x}{x^2} = 0$.
5. $\displaystyle x^2 \sin\left(\frac{1}{x}\right) + x^3 = o(x^4)$ *as* $x \longrightarrow +\infty$, *since* $\displaystyle\lim_{x \longrightarrow +\infty} \frac{x^2 \sin(\frac{1}{x}) + x^3}{x^4} = 0$.

Definition 2.7.2. *Let f and g be two functions defined in a neighborhood of x_0. We say that f is* ***equivalent*** *to g as $x \longrightarrow x_0$ if*

$$f - g = o(f),$$

and we write

$$f \underset{x_0}{\sim} g.$$

If there exists a neighborhood V of x_0 where f and g do not vanish, then this is equivalent to

$$f \underset{x_0}{\sim} g \Longleftrightarrow \lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = 1.$$

Example 2.7.2.

1. $e^x - 1 \sim x$ *as* $x \to 0$.

2. $\ln(1+x) \sim x$ *as* $x \to 0$.

3. $\sin x \sim x$ *as* $x \to 0$.

4. $1 - \cos x \sim \dfrac{x^2}{2}$ *as* $x \to 0$.

5. $\tan x \sim x$ *as* $x \to 0$.

6. $(1+x)^\alpha \sim 1 + \alpha x$ *as* $x \to 0$.

Property 2.7.1. *Let* f*,* f_1*,* g*, and* g_1 *be functions defined in a neighborhood of* x_0*. If* $f \underset{x_0}{\sim} f_1$ *and* $g \underset{x_0}{\sim} g_1$*, then:*

1. $\dfrac{f}{g} \underset{x_0}{\sim} \dfrac{f_1}{g_1}$*, where* g *and* g_1 *do not vanish near* x_0 *(except possibly at* x_0*);*

2. $fg \underset{x_0}{\sim} f_1 g_1$*;*

3. *If* $\lim\limits_{x \longrightarrow x_0} f(x) = \ell$*, then* $\lim\limits_{x \longrightarrow x_0} f_1(x) = \ell$*.*

Consequence:

When computing the limit of a product or a quotient, each factor may be replaced by a suitably chosen equivalent function to simplify the calculations. However, be careful: such replacements are NOT valid inside a sum or inside a composite function.

Example 2.7.3.

1) $x + 1 \underset{0}{\sim} 1$ *and* $-1 \underset{0}{\sim} -1$*, but*

$$x = x + 1 - 1 \underset{0}{\nsim} 0 = 1 - 1.$$

2) $x + 1 \underset{0}{\sim} x^2 + 1$ *but*

$$\ln(x+1) \underset{0}{\nsim} \ln(x^2+1),$$

since

$$\lim_{x \longrightarrow 0} \frac{\ln(x+1)}{\ln(x^2+1)} = \lim_{x \longrightarrow 0} \frac{x}{x^2} = \infty \neq 1.$$

Example 2.7.4. *Compute the following limits: ($\lambda \neq 0$)*

$$1)\ \lim_{x \longrightarrow 0} \frac{e^x - 1}{\sin x} \qquad 2)\ \lim_{x \longrightarrow 0} \frac{\sin^2 x}{1 - \cos x} \qquad 3)\ \lim_{x \longrightarrow 0} \frac{x(1-x)\sin \lambda x}{1 - \cos 2x} \qquad 4)\ \lim_{x \longrightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\tan^2 x}.$$

Remark 2.7.1. *There are cases where two functions are equivalent at x_0 but do not have a limit. For instance,*

$$f(x) = (x+1)\sin\left(\frac{1}{x}\right) \quad \textit{and} \quad g(x) = \sin\left(\frac{1}{x}\right).$$

We have $f \underset{0}{\sim} g$ even though neither f nor g has a (finite) limit as $x \longrightarrow 0$.

2.8 Practical Rules for Sum and Composition Equivalences

1. If f and g have the same sign, then $f + g \underset{x_0}{\sim} f_1 + g_1$, where $f \underset{x_0}{\sim} f_1$ and $g \underset{x_0}{\sim} g_1$.
2. If $f \underset{0}{\sim} a_1 x^{n_1}$ and $g \underset{0}{\sim} a_2 x^{n_2}$, then $f + g \underset{0}{\sim} a_n x^n$, where $n = \min\{n_1, n_2\}$. For example,
$$\sin^4 x + (1 - \cos x) \underset{0}{\sim} x^4 + \frac{x^2}{2} \underset{0}{\sim} \frac{x^2}{2}.$$
3. If $f \underset{\infty}{\sim} a_1 x^{n_1}$ and $g \underset{\infty}{\sim} a_2 x^{n_2}$, then $f + g \underset{\infty}{\sim} a_m x^m$, where $m = \max\{n_1, n_2\}$. For example,
$$x^5 + 3x^7 - x^3 + 5 \underset{\infty}{\sim} 3x^7.$$
4. If $n_1 = n_2 = n$, then $f + g \sim (a_{n_1} + a_{n_2})x^n$ at 0 or at ∞. For example,
$$\sin x + \frac{e^x - 1}{2} \underset{0}{\sim} x + \frac{1}{2}x \sim \frac{3}{2}x.$$
5. When $f = \circ(g)$, the function g dominates f, so $f + g \sim g$ as $x \longrightarrow x_0$.
For example,
$$x^3 \ln x + x \underset{+\infty}{\sim} x^3 \ln x$$
since $x = \circ(x^3 \ln x)$ at $+\infty$.
6. If $f \underset{x_0}{\sim} g$, then $f^\alpha \underset{x_0}{\sim} g^\alpha$.
7. If $f \underset{x_0}{\sim} g$ and $\lim_{x \longrightarrow x_0} f(x) \neq 1$, then $\ln f \underset{x_0}{\sim} \ln g$.
8. If $\lim_{x \longrightarrow x_0} [f(x) - g(x)] = 0$, then $e^f \underset{x_0}{\sim} e^g$.

Chapter 3

Differentiable Functions

3.1 Differentiable Functions

Definition 3.1.1. *Let I be an interval of $\mathbb{R}$, $x_0 \in I$, and let $f : I \longrightarrow \mathbb{R}$ be a function. We say that f is* ***differentiable*** *at x_0 if the limit*

$$\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is finite. This limit is called the ***derivative*** *of f at x_0 and is denoted by $f'(x_0)$. The function f is said to be differentiable on I if it is differentiable at every point of I. The mapping $x \longmapsto f'(x)$ is called the* ***derivative function****.*

Remark 3.1.1.
1) The above definition implicitly assumes that $x \to x_0$, but $x \neq x_0$.
2) The previous definition allows us to write

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + \epsilon(x) \quad \textit{with} \quad \lim_{x \longrightarrow x_0} \epsilon(x) = 0,$$

or equivalently,

$$f(x) - f(x_0) = (x - x_0)[f'(x_0) + \epsilon(x)].$$

Example 3.1.1.

1. $f(x) = c$, $\forall x \in \mathbb{R}$. *Let $x_0 \in \mathbb{R}$,*

$$f'(x_0) = \lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \longrightarrow x_0} \frac{c - c}{x - x_0} = 0 \Longrightarrow f'(x) = 0, \forall x \in \mathbb{R}.$$

2. $f(x) = x$, $\forall x \in \mathbb{R}$. *Let* $x_0 \in \mathbb{R}$,

$$f'(x_0) = \lim_{x \longrightarrow x_0} \frac{x - x_0}{x - x_0} = 1 \Longrightarrow f'(x) = 1, \forall x \in \mathbb{R}.$$

3. $f(x) = x^2$, $\forall x \in \mathbb{R}$. *Let* $x_0 \in \mathbb{R}$,

$$f'(x_0) = \lim_{x \longrightarrow x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \longrightarrow x_0} x + x_0 = 2x_0 \Longrightarrow f'(x) = 2x, \forall x \in \mathbb{R}.$$

4. $f(x) = \sin x$, $\forall x \in \mathbb{R}$. *Let* $x_0 \in \mathbb{R}$,

$$\begin{aligned} f'(x_0) &= \lim_{x \longrightarrow x_0} \frac{\sin x - \sin x_0}{x - x_0} = \lim_{x \longrightarrow x_0} \frac{2\sin(\frac{x-x_0}{2})\cos(\frac{x+x_0}{2})}{x - x_0} \\ &= \lim_{x \longrightarrow x_0} \frac{\sin(\frac{x-x_0}{2})}{\frac{x-x_0}{2}} \cos(\frac{x+x_0}{2}) = \cos x_0 \Longrightarrow f'(x) = \cos x, \forall x \in \mathbb{R}. \end{aligned}$$

Definition 3.1.2.

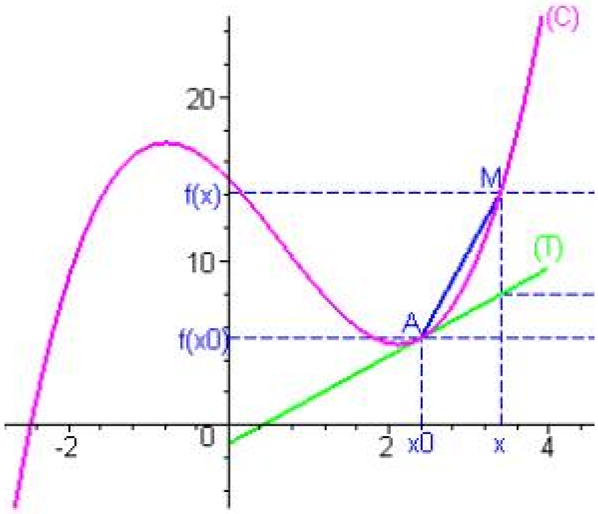
We can define the notion of the right-hand and left-hand derivatives at x_0 *in an analogous way:*

1. f *is* ***right-differentiable*** *at* x_0 *if* $\lim_{x \longrightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$ *exists, and it is then denoted by* $f'(x_0^+)$.

2. f *is* ***left-differentiable*** *at* x_0 *if* $\lim_{x \longrightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$ *exists, and it is then denoted by* $f'(x_0^-)$.

It is clear that f *is differentiable at* x_0 *if and only if* $f'(x_0^+) = f'(x_0^-)$.

3.2 Geometric Interpretation of the Derivative



Let $A(x_0, f(x_0))$ and $M(x, f(x))$ be two points on the curve (C) of the function f.
The expression

$$\frac{f(x) - f(x_0)}{x - x_0}$$

represents the slope of the line (AM).
To say that f is differentiable is to say that, as x approaches x_0 (that is, as M moves closer to A), the secant line (AM) tends to the tangent line (T) whose slope is $f'(x_0)$.
The line (T) is the tangent to (C) at the point $(x_0, f(x_0))$, and its equation is:

$$y = f'(x_0)(x - x_0) + f(x_0).$$

- If $f'(x_0) = 0$, then the tangent (T) is horizontal.
- If $f'(x_0) = \infty$, then the tangent (T) is vertical.
- If $f'(x_0^+) \neq f'(x_0^-)$, then (C) admits a right-hand and a left-hand tangent at x_0; that is, the graph has a **corner point** at $M_0(x_0, f(x_0))$.

Remark 3.2.1. *The differentiability of a function at a point can also be used to compute limits of rational functions that would otherwise give an indeterminate form.*

$$\lim_{x \to 1} \frac{\sin(\pi x)}{x - 1} = (\sin(\pi x))'_{x=1} = \pi \cos(\pi) = -\pi.$$

Example 3.2.1. *Study the differentiability of the following functions at the point x_0:*

1. $f(x) = |x-1|$ *at* $x_0 = 1$. *We have*
$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{x-1-0}{x-1} = 1,$$
and
$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{-x+1-0}{x-1} = -1.$$
*Thus, f is right- and left-differentiable at $x=1$, but $f'(1^+) = 1 \neq f'(1^-) = -1$. Therefore, f is **not differentiable** at $x=1$.*

2. $$f(x) = \begin{cases} \sin x, & x > 0, \\ x, & x \leq 0 \end{cases} \quad \textit{at } x_0 = 0.$$
We have
$$\lim_{x\to 0^+} \frac{\sin x - \sin 0}{x-0} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 = \lim_{x\to 0^-} \frac{x-0}{x-0}.$$
Hence, f is differentiable at 0 and $f'(0) = 1$.

3. $$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0 \end{cases} \quad \textit{at } x_0 = 0.$$
We have
$$\lim_{x\to 0} \frac{x\sin(\frac{1}{x}) - 0}{x-0} = \lim_{x\to 0} \sin\left(\frac{1}{x}\right) \quad \textit{does not exist.}$$
*Therefore, f is **not differentiable** at 0.*

3.3 Properties and computation of derivatives

Theorem 3.3.1. *If f is differentiable at x_0, then f is continuous at x_0.*

Proof. Since f is differentiable at x_0, we have
$$f(x) - f(x_0) = (x-x_0)\big(f'(x_0) + \varepsilon(x)\big), \quad \text{where} \quad \lim_{x\to x_0} \varepsilon(x) = 0.$$
Hence,
$$\lim_{x\to x_0} [f(x) - f(x_0)] = \lim_{x\to x_0} (x-x_0) \lim_{x\to x_0} (f'(x_0) + \varepsilon(x)) = 0.$$
Therefore, f is continuous at x_0.

Remark 3.3.1.

- *By contrapositive, we obtain:*
  «If f is not continuous at x_0, then it is not differentiable at x_0.»

- *The converse of the theorem is false. Indeed,*
  $f(x) = |x|$ is continuous at 0 since $\lim_{x \longrightarrow 0} f(x) = 0$, but it is not differentiable at 0 because
  $$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x}$$
  has no limit as $x \to 0$.

Theorem 3.3.2. *Let f and g be differentiable functions on an interval I. Then:*

1. *For every $\alpha \in \mathbb{R}$, αf is differentiable on I, and $(\alpha f)' = \alpha f'$.*

2. *$f + g$ is differentiable on I, and $(f + g)' = f' + g'$.*

3. *fg is differentiable on I, and $(fg)' = f'g + fg'$.*

4. *$\dfrac{f}{g}$ is differentiable at every point x_0 in I such that $g(x_0) \neq 0$, and*
   $$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Theorem 3.3.3. *If f is differentiable at x_0 and g is differentiable at $y_0 = f(x_0)$, then the composite function $g \circ f$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = f'(x_0)\, g'(f(x_0)).$$

Example 3.3.1.

1. $\left(\sin x^2\right)' = 2x \cos x^2$.

2. $\left(\ln \sqrt{x}\right)' = \dfrac{1}{2\sqrt{x}} \dfrac{1}{\sqrt{x}} = \dfrac{1}{2x}$.

3. $\left(\cos(x^3 + 2x + 1)\right)' = -(3x^2 + 2)\sin(x^3 + 2x + 1)$.

Exercise 3.3.1. *Compute the derivative of the following functions:*

1) $f(x) = \sqrt{x^2 - 1}$ 2) $f(x) = \sqrt{x + \sqrt{x+2}}$ 3) $f(x) = \sqrt{\ln x + 1} + \ln(\sqrt{x} + 1)$

4) $f(x) = e^{\cos\sqrt{x}}$ 5) $f(x) = (x^3 + 7x - 1)^{2018}$ 6) $f(x) = \dfrac{\sqrt{\cos x}}{1 - e^{-x}}$.

Theorem 3.3.4 (Derivative of the Inverse Function)**.**
Let f be a bijective and continuous function on an interval I, with range $J = f(I)$. Assume that f is differentiable at some point $x_0 \in I$ and that $f'(x_0) \neq 0$. Then the inverse function $f^{-1} : J \longrightarrow I$ is differentiable at $y_0 = f(x_0)$, and its derivative is given by

$$(f^{-1})'(y_0) = (f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}.$$

3.4 Application to the Inverse Trigonometric Functions

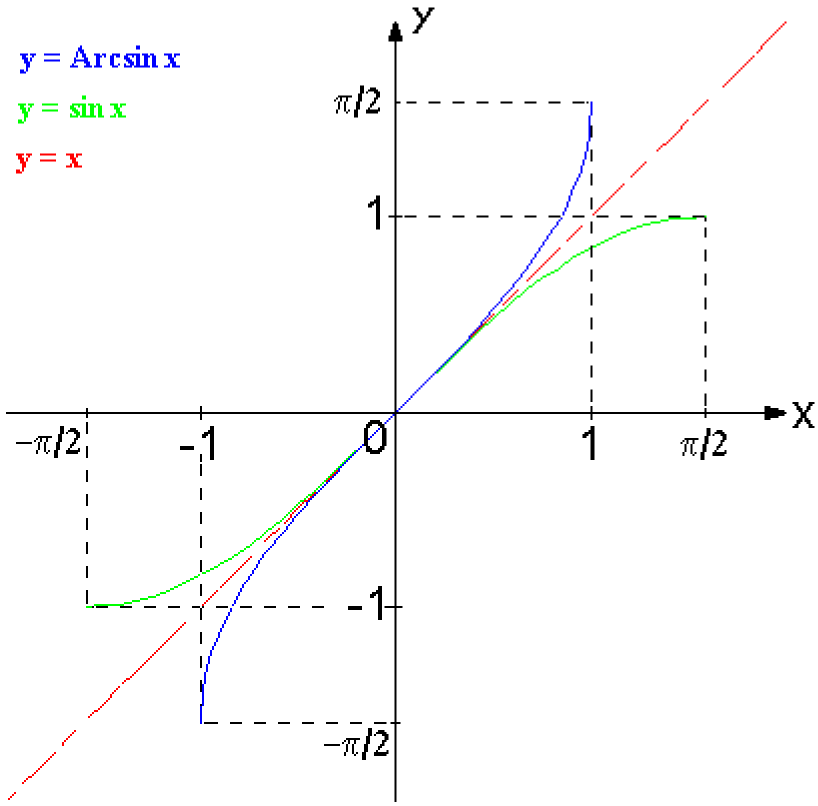
Definition 3.4.1 (**The function** arcsin)**.**
The function $\sin x$ *is continuous and differentiable on* $\big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$ *and its derivative is* $(\sin x)' = \cos x \geq 0$ *on this interval.*

x	$-\frac{\pi}{2}$ $\qquad$ $\frac{\pi}{2}$
$\cos x$	$+$
$\sin x$	$-1 \nearrow 1$

Therefore, the function $\sin x$ *is strictly increasing on* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$*, and hence defines a bijection from* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *onto* $[-1, 1]$*. The inverse function, denoted by*

$$\begin{aligned} \arcsin = \sin^{-1} : [-1,1] &\longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}] \\ y &\longmapsto \arcsin y, \end{aligned}$$

is called the arcsine function.



It is clear that the function arcsin is continuous, odd, and strictly increasing on $[-1, 1]$. Moreover,

1. $\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$: $\arcsin(\sin x) = x$.
2. $\forall y \in [-1, 1]$: $\sin(\arcsin y) = y$.
3. $\forall y \in]-1, 1[$: $(\arcsin y)' = \dfrac{1}{\sqrt{1-y^2}}$.

Let us prove 3. Let $y \in]-1, 1[$. We have $(\sin x)' = \cos x > 0$ on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$; therefore, the inverse function arcsin is differentiable on $]-1, 1[$, and

$$(\arcsin y)' = \frac{1}{(\sin x)'} = \frac{1}{\cos x}, \quad \text{where } y = \sin x,\ y \in]-1, 1[,\ x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

Hence,

$$(\arcsin y)' = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}.$$

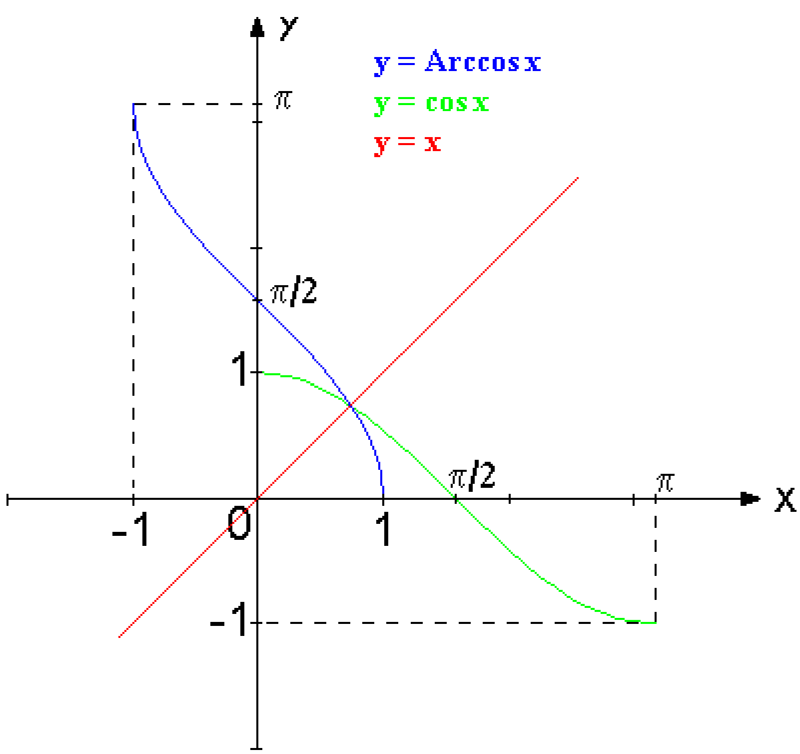
Definition 3.4.2 (**of the function** arccos)**.**
The function $\cos x$ *is continuous and differentiable on* $[0, \pi]$*, with derivative* $(\cos x)' = -\sin x \leq 0$.

x	$0 \qquad \pi$
$-\sin x$	$-$
$\cos x$	$1 \searrow -1$

Thus, the function cos *is strictly decreasing on* $[0, \pi]$*, and therefore defines a bijection from* $[0, \pi]$ *onto* $[-1, 1]$*. Its inverse function is denoted by:*

$$\begin{aligned} \arccos = \cos^{-1} : [-1, 1] &\longrightarrow [0, \pi] \\ y &\longmapsto \arccos y, \end{aligned}$$

is called the arccosine function.



It is clear that the function arccos is continuous and strictly increasing on $[-1, 1]$. Moreover,

1. $\forall x \in [0, \pi] : \arccos(\cos x) = x$.
2. $\forall y \in [-1, 1] : \cos(\arccos y) = y$.
3. $\forall y \in]-1, 1[$: $(\arccos y)' = \dfrac{-1}{\sqrt{1-y^2}}$.

Let us prove 3. Let $y \in]-1, 1[$. We have $(\cos x)' = -\sin x < 0$ on $\big]0, \pi\big[$; therefore, the inverse function arccos is differentiable on $]-1, 1[$, and

$$(\arccos y)' = \frac{1}{(\cos x)'} = \frac{-1}{\sin x}, \quad \text{where } y = \cos x,\ y \in]-1, 1[,\ x \in \big]0, \pi\big[.$$

Hence,

$$(\arccos y)' = \frac{-1}{\sqrt{1-\cos^2 x}} = \frac{-1}{\sqrt{1-y^2}}.$$

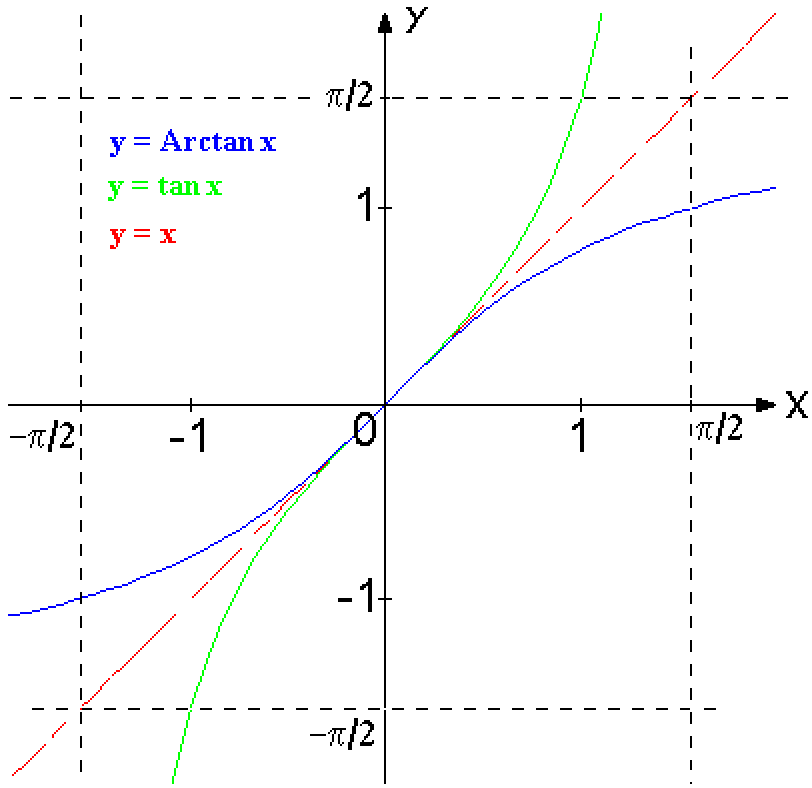
Definition 3.4.3 (of the function arctan**).**
The function $\tan x$ *is continuous and differentiable on* $\big]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\big[$, *with derivative* $(\tan x)' = \dfrac{1}{\cos^2 x} > 0$.

x	$-\frac{\pi}{2}$		$\frac{\pi}{2}$
$\frac{1}{\cos^2 x}$		$+$	
$\tan x$	$-\infty$	$\nearrow$	$+\infty$

Thus, the function tan *is strictly increasing on* $\big]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\big[$, *and therefore defines a bijection from* $\big]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\big[$ *onto* $\mathbb{R}$. *Its inverse function is denoted by:*

$$\begin{aligned} \arctan = \tan^{-1} : \mathbb{R} &\longrightarrow \Big]-\frac{\pi}{2}, \frac{\pi}{2}\Big[\\ y &\longmapsto \arctan y, \end{aligned}$$

is called the arctangent function.



It is clear that the function arctan is continuous, odd, and strictly increasing on $\mathbb{R}$. Moreover,

1. $\forall x \in \left] -\dfrac{\pi}{2}, \dfrac{\pi}{2} \right[: \arctan(\tan x) = x.$
2. $\forall y \in \mathbb{R} : \tan(\arctan y) = y.$
3. $\forall y \in \mathbb{R} : (\arctan y)' = \dfrac{1}{1+y^2}.$

Let us prove 3. Let $y \in \mathbb{R}$. We have $(\tan x)' = 1 + \tan^2 x > 0$ on $\left] -\dfrac{\pi}{2}, \dfrac{\pi}{2} \right[$; therefore, the inverse function arctan is differentiable on $\mathbb{R}$, and

$$(\arctan y)' = \frac{1}{(\tan x)'} = \frac{1}{1+\tan^2 x}, \quad \text{where } y = \tan x,\ y \in \mathbb{R},\ x \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[.$$

Hence,

$$(\arctan y)' = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}.$$

Remark 3.4.1.

$$\begin{aligned} arccotan = cotan^{-1} : \mathbb{R} &\longrightarrow]0, \pi[\\ y &\longmapsto arccotan y = x \Longleftrightarrow y = \tan x. \end{aligned}$$

3.5 Hyperbolic Functions and Their Inverses

Hyperbolic Functions

Definition 3.5.1. *The real functions*

- $x \longmapsto \cosh x = \dfrac{e^x + e^{-x}}{2}$ *is called the* hyperbolic cosine.
- $x \longmapsto \sinh x = \dfrac{e^x - e^{-x}}{2}$ *is called the* hyperbolic sine.
- $x \longmapsto \tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^{2x} - 1}{e^{2x} + 1}$ *is called the* hyperbolic tangent.
- $x \longmapsto \coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^{2x} + 1}{e^{2x} - 1}$, $x \neq 0$, *is called the* hyperbolic cotangent.

From this definition, we deduce

$$\sinh(0) = 0, \quad \cosh(0) = 1, \quad \tanh(0) = 0, \quad \cosh(-x) = \cosh x, \quad \sinh(-x) = -\sinh x.$$

Propertys 3.5.1.

1. $\cosh^2 x - \sinh^2 x = 1$.
2. $\dfrac{1}{\cosh^2 x} = 1 - \tanh^2 x$.
3. *The functions* sinh, cosh, *and* tanh *are differentiable on* $\mathbb{R}$ *with*

$$(\cosh x)' = \sinh x, \quad (\sinh x)' = \cosh x, \quad (\tanh x)' = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x,$$

and the function coth *is differentiable on* $\mathbb{R} \setminus \{0\}$ *with*

$$(\coth x)' = -\frac{1}{\sinh^2 x}.$$

Inverse Hyperbolic Functions

Definition 3.5.2 (**of the function** arsinh)**.**
Let us consider the function sinh *on* $\mathbb{R}$*. The function* sinh *is continuous and differentiable on* $\mathbb{R}$*, with derivative* $(\sinh x)' = \cosh x > 0$*.*

x	$-\infty$		$+\infty$
$\cosh x$		$+$	
$\sinh x$	$-\infty$	$\nearrow$	$+\infty$

The function sinh *is continuous and strictly increasing on* $\mathbb{R}$*, and therefore defines a bijection from* $\mathbb{R}$ *onto* $\mathbb{R}$*. Its inverse function is defined and denoted by*

$$\begin{aligned} \sinh^{-1} = \operatorname{arsinh} : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \operatorname{arsinh} x = \ln\bigl(x + \sqrt{1+x^2}\bigr). \end{aligned}$$

The function arsinh *is differentiable on* $\mathbb{R}$*, with derivative*

$$(\operatorname{arsinh} x)' = \frac{1}{\sqrt{1+x^2}}.$$

Definition 3.5.3 (**of the function** arcosh)**.**
Let us consider the function cosh *on* $\mathbb{R}$*. The function* cosh *is continuous and differentiable on* $\mathbb{R}$*, with derivative* $(\cosh x)' = \sinh x$*.*

x	$-\infty$		0		$+\infty$
sinh		$-$		$+$	
cosh	$+\infty$	$\searrow$	1	$\nearrow$	$-\infty$

We will focus our study on $[0, +\infty[$*. The function* cosh *is continuous and strictly increasing on* $[0, +\infty[$*, and therefore defines a bijection from* $[0, +\infty[$ *onto* $[1, +\infty[$*. Its inverse* *function is defined and denoted by*

$$\begin{aligned}\cosh^{-1} = \operatorname{arcosh} : [1, +\infty[&\longrightarrow [0, +\infty[\\ x &\longmapsto \operatorname{arcosh} x = \ln\bigl(x + \sqrt{x^2 - 1}\bigr).\end{aligned}$$

The function arcosh *is differentiable on* $(1, +\infty[$, *with derivative*

$$(\operatorname{arcosh} x)' = \frac{1}{\sqrt{x^2 - 1}}.$$

Definition 3.5.4 **(of the function** artanh**).**
Let us consider the function tanh *on* $\mathbb{R}$. *The function* tanh *is continuous and differentiable on* $\mathbb{R}$, *with derivative* $(\tanh x)' = \dfrac{1}{(\cosh^2 x)} > 0$.

x	$-\infty$ $\qquad$ $+\infty$
$\dfrac{1}{\cosh^2 x}$	$+$
$\tanh x$	$-1 \nearrow 1$

The function tanh *is continuous and strictly increasing on* $\mathbb{R}$, *and therefore defines a bijection from* $\mathbb{R}$ *onto* $]-1, 1[$. *Its inverse function is defined and denoted by:*

$$\begin{aligned}\tanh^{-1} = \operatorname{artanh} :]-1, 1[&\longrightarrow \mathbb{R} \\ x &\longmapsto \operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).\end{aligned}$$

The function artanh is differentiable on $]-1, 1[$, with derivative

$$(\operatorname{artanh} x)' = \frac{1}{1 - x^2}.$$

3.6 Higher-Order Derivatives

Let f be a function defined on an interval I. If the derivative f' itself has a derivative, this is called the second derivative or the derivative of order 2 of f, and is denoted by f''. In this case, f' is called the first derivative of f. Similarly, we can define the derivative of order n (or the n-th derivative) of f, denoted by $f^{(n)}$, as the derivative of $f^{(n-1)}$. We adopt the convention that the derivative of order 0 is the function itself, $f^{(0)} = f$.

Remark 3.6.1. *We can also write $f'(x)$ as $\dfrac{df}{dx}$ and $f^{(n)}(x)$ as $\dfrac{d^n f}{dx^n}$.*

Example 3.6.1. *Compute the derivative of order n of:*

1. $f(x) = e^{ax} \implies f'(x) = ae^{ax} \implies f''(x) = a^2e^{ax} \ldots \implies f^{(n)}(x) = a^n e^{ax}$.

2. $g(x) = \dfrac{1}{ax+b} \implies g'(x) = \dfrac{-a}{(ax+b)^2} \implies g''(x) = \dfrac{2a^2}{(ax+b)^3}$
   $g^{(3)}(x) = \dfrac{-6a^3}{(ax+b)^4} \implies g^{(4)}(x) = \dfrac{24a^4}{(ax+b)^5} \ldots \implies g^{(n)}(x) = \dfrac{(-1)^n n! a^n}{(ax+b)^{n+1}}$, $\forall n \in \mathbb{N}$.

3.7 Functions of Class C^n

A function f defined on an interval I is said to be of class $C^n(I)$ if it is n times differentiable and its n-th derivative $f^{(n)}$ is continuous on I. In particular, f is of class $C^0(I)$ if it is continuous, and of class $C^\infty(I)$ if it is infinitely differentiable.

Example 3.7.1. *Determine the differentiability class of the following functions:*

1. $f(x) = x^2$. *We have*

   $$f'(x) = 2x, \quad f''(x) = 2, \quad f^{(n)}(x) = 0 \text{ for } n \geq 3,$$

   all of which are continuous. Hence, $f \in C^\infty(\mathbb{R})$.

2. $g(x) = \sin x$. *For all $n \in \mathbb{N}$,*

   $$g^{(n)}(x) = \sin\Big(x + \frac{n\pi}{2}\Big)$$

   is continuous. Therefore, $g \in C^\infty(\mathbb{R})$.

3. $$h(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

   (a) *The function h is defined on $\mathbb{R}$, and the function*

   $$x \longmapsto x^2 \sin\big(\frac{1}{x}\big)$$

is differentiable on $\mathbb{R}^*$ *since it is the product of two differentiable functions on* $\mathbb{R}^*$*:* $x \longmapsto x^2$ *and* $x \longmapsto \sin\left(\dfrac{1}{x}\right)$*. Moreover,* $x \longmapsto \sin\left(\dfrac{1}{x}\right)$ *is the composition of two differentiable functions on* $\mathbb{R}^*$*:* $x \longmapsto \dfrac{1}{x}$ *and* $x \longmapsto \sin x$*.*
At $x = 0$*, the derivative exists since*

$$\lim_{x \longrightarrow 0} \frac{h(x) - h(0)}{x} = \lim_{x \longrightarrow 0} x \sin\left(\frac{1}{x}\right) = 0.$$

because $\sin\left(\dfrac{1}{x}\right)$ *is bounded (*$\left|\sin\left(\dfrac{1}{x}\right)\right| \leq 1$*) and* $x \longrightarrow 0$*. Therefore,* h *is differentiable at* 0*, and hence differentiable on* $\mathbb{R}$*.*

(b) However, the derivative

$$h'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is not continuous at $x = 0$ *because*

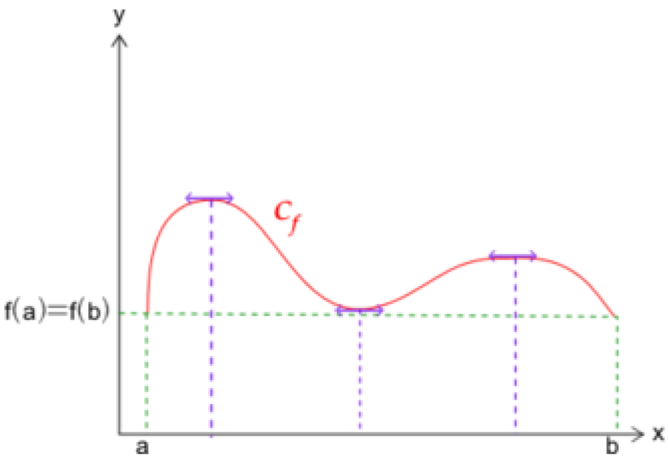
$$\lim_{x \longrightarrow 0}\left(2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right) \text{ does not exist.}$$

Hence, $h \notin C^1(\mathbb{R})$*, though* $h \in C(\mathbb{R})$*.*

3.8 Rolle's Theorem

Theorem 3.8.1.
Let $f : [a, b] \longrightarrow \mathbb{R}$ *be a function continuous on* $[a, b]$*, differentiable on* $]a, b[$*, and such that* $f(a) = f(b)$*. Then there exists a number* $c \in]a, b[$ *such that* $f'(c) = 0$*.*

Geometric Interpretation



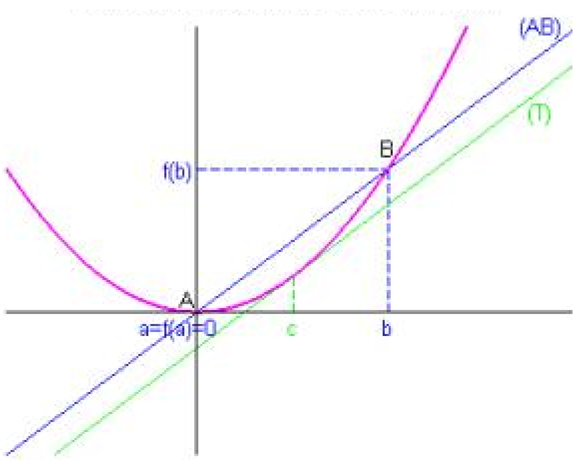
Let $A = (a, f(a))$ and $B = (b, f(b))$. Rolle's Theorem states that on the segment AB of the graph, corresponding to the interval $[a, b]$, there exists at least one point $C = (c, f(c))$, distinct from A and B, at which the tangent to the curve is parallel to the line segment AB.

3.9 Mean Value Theorem

Theorem 3.9.1. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$. Then there exists at least one number $c \in]a, b[$ such that*

$$f(b) - f(a) = (b - a)f'(c).$$

Geometric Interpretation



Consider the graph of f and the points $(a, f(a))$ and $(b, f(b))$. The line AB has slope

$$p = \frac{f(b) - f(a)}{b - a}$$

and equation

$$y = d(x) = f(a) + p(x - a).$$

Define

$$R(x) = f(x) - d(x).$$

Clearly, R is continuous on $[a, b]$ and differentiable on $]a, b[$, and moreover $R(a) = R(b) = 0$. By Rolle's Theorem, there exists a number $c \in]a, b[$ such that $R'(c) = 0$. But

$$R'(x) = f'(x) - p,$$

so that

$$f'(c) = p = \frac{f(b) - f(a)}{b - a}.$$

This equality shows that at $x = c$, the tangent line to the curve is parallel to the line segment AB.

Corollary 3.9.1. *Let $f : I \to \mathbb{R}$ be a differentiable function (with I any interval). Then, for any two distinct points $x_1, x_2 \in I$, there exists at least one number $c \in]x_1, x_2[$ such that*

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c).$$

Example 3.9.1.

1) Apply the Mean Value Theorem to the function $f(t) = \ln(1+t)$ *on the interval* $[0, x]$.

2) Deduce that $\ln(1+x) < x$ *for all* $x > 0$.

Solution 3.9.1.

1) The function f *is continuous on* $[0, x]$ *and differentiable on* $]0, x[$*, so by the Mean Value Theorem, there exists* $c \in]0, x[$ *such that*

$$f(x) - f(0) = xf'(c) \quad \Longleftrightarrow \quad \ln(1+x) = \frac{x}{1+c}.$$

2) Since $c > 0$*, we have* $1 + c > 1$*, hence* $\frac{1}{1+c} < 1$*, which gives*

$$\ln(1+x) = \frac{x}{1+c} < x.$$

3.10 L'Hôpital's Rule

Let f and g be two functions defined and differentiable in a neighborhood of x_0, with g' nonzero in this neighborhood.
If $\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, with $x_0 \in \bar{\mathbb{R}}$, and

$$\lim_{x \longrightarrow x_0} \frac{f'(x)}{g'(x)} = l,$$

then

$$\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = l.$$

Exercise 3.10.1. *Compute the following limits:*

1) $\lim_{x \longrightarrow 0} \frac{1 - \cos x^2}{x^4}$ 2) $\lim_{x \longrightarrow x_0} \frac{\cos x - \cos x_0}{\sin x - \sin x_0}$ 3) $\lim_{x \longrightarrow +\infty} \frac{\ln x}{e^x}$ 4) $\lim_{x \longrightarrow 0} \frac{\sin x - x}{x(\cos x - 1)}$.

Solution 3.10.1.

1) $\displaystyle \lim_{x\longrightarrow 0}\frac{1-\cos x^2}{x^4}\overset{H}{=}\lim_{x\longrightarrow 0}\frac{2x\sin x^2}{4x^3}=\lim_{x\longrightarrow 0}\frac{\sin x^2}{2x^2}=\frac{1}{2}.$

2) $\displaystyle \lim_{x\longrightarrow x_0}\frac{\cos x-\cos x_0}{\sin x-\sin x_0}\overset{H}{=}\lim_{x\longrightarrow x_0}\frac{-\sin x}{\cos x}=-\tan x_0.$

3) $\displaystyle \lim_{x\longrightarrow +\infty}\frac{\ln x}{e^x}\overset{H}{=}\lim_{x\longrightarrow +\infty}\frac{1}{xe^x}=0.$

4)

$$\begin{aligned}\lim_{x\longrightarrow 0}\frac{\sin x-x}{x(\cos x-1)} &\overset{H}{=} \lim_{x\longrightarrow 0}\frac{\cos x-1}{\cos x-1-x\sin x}\overset{H}{=}\lim_{x\longrightarrow 0}\frac{-\sin x}{-2\sin x-x\cos x}\\ &\overset{H}{=} \lim_{x\longrightarrow 0}\frac{-\cos x}{-3\cos x+x\sin x}=\frac{1}{3}.\end{aligned}$$

3.11 Extrema

Definition 3.11.1. *Let f be a function defined on $I\subset\mathbb{R}$ with values in $\mathbb{R}$. The function f has a* maximum *(respectively a* minimum*) at $x_0\in I$ if*

$$\forall x\in I,\ f(x)\le f(x_0)\quad (\textit{respectively}\ \forall x\in I,\ f(x)\ge f(x_0)).$$

An *extremum* is either a maximum or a minimum. Every extremum is a local extremum.

Theorem 3.11.1. *Let f be a C^2 function on I, and let $x_0\in I$.*

1) *If $f'(x_0)=0$ and $f''(x_0)<0$, then x_0 is a local maximum.*

2) *If $f'(x_0)=0$ and $f''(x_0)>0$, then x_0 is a local minimum.*

Example 3.11.1.

1) *The function $f(x)=x^2$ defined on $\mathbb{R}$ has a global minimum at 0.*

2) *The function $f(x)=\cos x$ defined on $\mathbb{R}$ has a local maximum at 0.*

Chapter 4

Integrals

4.1 Antiderivatives

Definition 4.1.1. *Let f be a function defined and continuous on an interval I. A function $F : I \to \mathbb{R}$ is called an* antiderivative *of f if*

$$F'(x) = f(x) \quad \textit{for all } x \in I.$$

Remark 4.1.1. *This definition implies that F is differentiable.*

For example, the function

$$x \longmapsto F(x) = \frac{1}{3}x^3$$

is an antiderivative of

$$x \longmapsto f(x) = x^2$$

on $\mathbb{R}$, since

$$F'(x) = \left(\frac{1}{3}x^3\right)' = x^2 \quad \text{for all } x \in \mathbb{R}.$$

It is easy to see that the functions

$$x \longmapsto F(x) = \frac{1}{3}x^3 + C$$

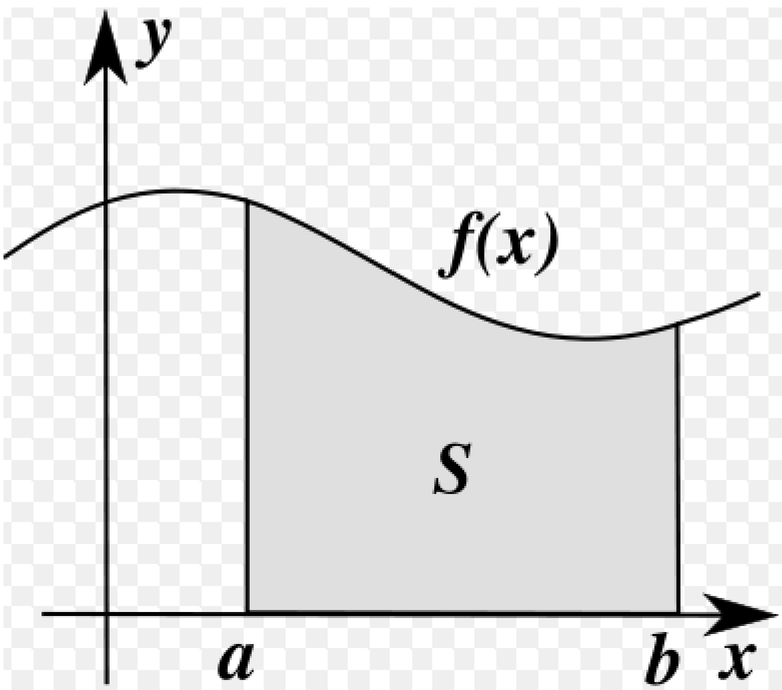
(where C is an arbitrary constant) are also antiderivatives of $f(x) = x^2$.

4.2 Definite Integrals

Definition 4.2.1. *Let f be a continuous function on $[a, b]$, and let F be an antiderivative of f. The number*

$$S = F(b) - F(a)$$

is called the definite integral *of f from a to b, and represents the area bounded by the graph of f (the curve C_f), the x-axis ($y = 0$), and the vertical lines $x = a$ and $x = b$.*



Notation:

$$S = \int_a^b f(x)\,dx = F(b) - F(a) = [F(x)]_a^b\,,$$

and is read as: the integral of $f(x)$ from a to b with respect to x.

Definition 4.2.2. *The quantity*

$$\int_a^b f(x)\,dx$$

is called the definite integral *of the function f over $[a, b]$, and we say that f is* Riemann integrable.

- *The number a is the* lower limit *of the integral and b the* upper limit.

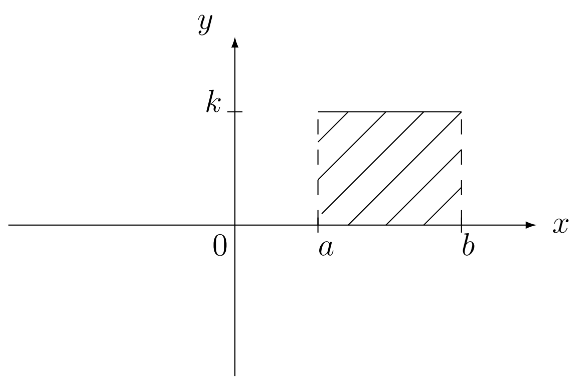
- $[a,b]$ *is the* interval of integration.
- x *is the* variable of integration*, so that*

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$$

x is called a dummy variable *and plays no role in the definition; that is, the definite integral depends only on the function f and the limits, but not on the variable of integration.*

Example 4.2.1. *Compute the area under $f(x) = k$ on $[a,b]$:*

$$\int_a^b f(x)\,dx = k(b-a).$$



4.3 Properties of the Riemann Integral

4.3.1 Linearity of the Integral with Respect to the Function

Let f and g be two integrable functions on $[a,b]$. Then $(f+g)$ and (αf) are also integrable on $[a,b]$, and we have:

1. $\displaystyle\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$, where $\alpha \in \mathbb{R}$.
2. $\displaystyle\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

4.3.2 Additivity (Chasles' Relation)

If f is integrable on $[a,b]$, then for any $c \in [a,b]$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Also,

$$\int_a^a f(x)\,dx = 0 \quad \Longrightarrow \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

4.3.3 Order Property

Suppose f is integrable on $[a,b]$ with $a \le b$. Then:

$$f(x) \ge 0 \text{ for all } x \in [a,b] \implies \int_a^b f(x)\,dx \ge 0.$$

If

$$f(x) \ge g(x) \text{ on } [a,b],$$

applying the previous property to $f(x) - g(x) \ge 0$, we obtain

$$\int_a^b \big(f(x) - g(x)\big)\,dx \ge 0,$$

and by linearity,

$$\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$$

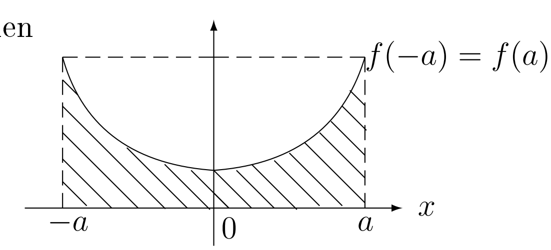
Moreover,

$$\begin{aligned} -|f(x)| \le f(x) \le |f(x)| &\implies -\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx \\ &\implies \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx. \end{aligned}$$

4.3.4 Symmetry

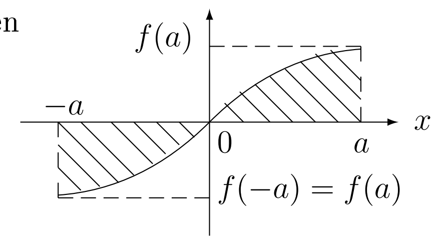
1. If the function f is even and integrable, then

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$



2. If the function f is odd and integrable, then

$$\int_{-a}^{a} f(x)dx = 0.$$



4.4 Indefinite Integrals

The set of all antiderivatives of a function $f : [a, b] \to \mathbb{R}$ is called the *indefinite integral* of f, denoted by

$$\int f(x)\,dx = F(x) + C,$$

where F is an antiderivative of f and $C \in \mathbb{R}$.
From the definition of the integral, we have

$$\left(\int f(x)\,dx\right)' = (F(x) + C)' = f(x).$$

Propertys 4.4.1.
If f and g admit antiderivatives, then so do $f + g$, and we have:

1. $\displaystyle\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx.$
2. $\displaystyle\int \alpha f(x)\,dx = \alpha \int f(x)\,dx$, *for any* $\alpha \in \mathbb{R}$.
3. *If* $\displaystyle\int f(x)\,dx = F(x) + C$, *then* $\displaystyle\int f(ax + b)\,dx = \frac{1}{a}F(ax + b) + C, \quad C \in \mathbb{R}.$

Theorem 4.4.1.
Every continuous function $f : [a, b] \to \mathbb{R}$ admits an antiderivative, which can be defined by

$$F(x) = \int_{c}^{x} f(t)\,dt, \quad F(c) = 0,$$

where $c \in [a, b]$ is fixed.

Theorem 4.4.2 (Derivative of an Integral with Variable Limits)**.**
Let f be continuous on $[a,b]$, and let $u, v : [\alpha, \beta] \to [a,b]$ be two C^1 functions. Define

$$g(x) = \int_{u(x)}^{v(x)} f(t)\, dt.$$

Then g is differentiable and

$$g'(x) = f(v(x))\, v'(x) - f(u(x))\, u'(x).$$

Proof. Since f is continuous, it admits an antiderivative F, so that

$$\begin{aligned} g(x) = \int_{u(x)}^{v(x)} f(t)\, dt &= \int_{c}^{v(x)} f(t)\, dt + \int_{u(x)}^{c} f(t)\, dt \\ &= \int_{c}^{v(x)} f(t)\, dt - \int_{c}^{u(x)} f(t)\, dt \\ &= F(v(x)) - F(u(x)). \end{aligned}$$

Differentiating, we obtain

$$g'(x) = F'(v(x))\, v'(x) - F'(u(x))\, u'(x) = f(v(x))\, v'(x) - f(u(x))\, u'(x).$$

Example 4.4.1. $g(x) = \int_{x}^{2x} \sqrt{1+t^4} dt = \int_{x}^{2x} f(t) dt$ *and its derivative is*

$$g'(x) = 2\sqrt{1+(2x)^4} - \sqrt{1+x^4}.$$

Exercise 4.4.1. *Calculate the following indefinite integrals:*

1) $\int x dx$ 2) $\int x^2 dx$ 3) $\int x^n dx, n \neq -1$ 4) $\int \frac{dx}{(x-a)^n}, n \neq 1$

5) $\int \frac{1}{x} dx$ 6) $\int e^x dx$ 7) $\int e^{2x} dx$ 8) $\int \sqrt{x} dx$

9) $\int \sin x dx$ 10) $\int \cos x dx$ 11) $\int \frac{1}{\cos^2 x} dx$ 12) $\int \tan x dx$

13) $\int shx dx$ 14) $\int chx dx$ 15) $\int \frac{1}{1-x^2}dx = \frac{1}{2}\ln\frac{|1+x|}{|1-x|}$

16) $\int thx dx$ 17) $\int \tan^2 x dx$ 18) $\int \frac{1}{\sqrt{x^2-1}}dx = \ln(x+\sqrt{x^2-1})$

19) $\int \frac{1}{1+x^2}dx$ 20) $\int \frac{1}{\sqrt{1-x^2}}dx.$

Examples 4.4.1.

$$\begin{aligned}
\int (2x^3 - 5\cos x + 3\sqrt{x})dx &= 2\int x^3 dx - 5\int \cos x dx + 3\int \sqrt{x}dx \\
&= \frac{x^4}{2} - 5\sin x + 2x\sqrt{x} + c \ (c \in \mathbb{R}). \\
\int \cos 7x dx &= \frac{1}{7}\sin 7x + c \ (c \in \mathbb{R}). \\
\int \frac{1}{e^{2x+3}}dx &= \frac{-1}{2}\frac{1}{e^{2x+3}} + c \ (c \in \mathbb{R}). \\
\int_0^1 \frac{x}{\sqrt{1+x^2}} &= [\sqrt{1+x^2}]_0^1 = \sqrt{2} - 1.
\end{aligned}$$

Before concluding this section, let us recall the following integrals:

$$\begin{aligned}
\int f'(x) f^n(x) dx &= \frac{1}{n+1} f^{n+1}(x) + c, \qquad n \neq -1, \\
\int \frac{f'(x)}{f(x)}dx &= \ln \mid f(x) \mid + c, \qquad n = -1.
\end{aligned}$$

4.5 Integration Methods

4.5.1 Integration by Parts

Theorem 4.5.1. *Let u and v be two C^1 functions on $[a,b]$. Then*

$$\int_a^b u(x)v'(x)\,dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b u'(x)v(x)\,dx,$$

where $\Big[u(x)v(x)\Big]_a^b = u(b)v(b) - u(a)v(a).$

Proof. By integrating the derivative of the product $(uv)' = u'v + uv'$, we have

$$\begin{aligned}\int_a^b (uv)'(x)\,dx &= \Big[u(x)v(x)\Big]_a^b \\ &= \int_a^b u(x)v'(x)\,dx + \int_a^b u'(x)v(x)\,dx,\end{aligned}$$

which immediately implies

$$\int_a^b u(x)v'(x)\,dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b u'(x)v(x)\,dx.$$

Examples 4.5.1.

1. $\displaystyle\int xe^x dx = \int x(e^x)'dx = xe^x - \int (x)'e^x dx = xe^x - \int e^x dx = (x-1)e^x + c.$

2. $I = \displaystyle\int x^2 \sin x dx$. *Applying integration by parts twice, we obtain*

$$\begin{aligned}\int x^2 \sin x dx = \int x^2(-\cos x)'dx &= -x^2\cos x + \int 2x\cos x dx \\ &= -x^2\cos x + \int 2x(\sin x)'dx,\end{aligned}$$

then

$$\begin{aligned}\int x^2 \sin x dx &= -x^2\cos x + 2x\sin x - 2\int \sin x dx \\ &= -x^2\cos x + 2x\sin x + 2\cos x + c,\end{aligned}$$

hence

$$\int x^2 \sin x dx = (2-x^2)\cos x + 2x\sin x + c.$$

3. $I = \displaystyle\int_0^1 \arctan x dx$, *we set*

$$\begin{cases} u(x) = \arctan x & v'(x) = 1 \\ u'(x) = \dfrac{1}{1+x^2} & v(x) = x \end{cases}$$

then

$$\begin{aligned}I = \Big[x\arctan x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}dx &= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}dx \\ &= \frac{\pi}{4} - \frac{1}{2}\Big[\ln(1+x^2)\Big]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln(2).\end{aligned}$$

4. $I = \displaystyle\int \ln x dx$, *we set* $\begin{cases} u(x) = \ln x & v'(x) = 1 \\ u'(x) = \dfrac{1}{x} & v(x) = x \end{cases}$

thus

$$I = x \ln x - \int dx = x \ln x - x + c.$$

5. $I = \displaystyle\int \arccos x dx$, *we set* $\begin{cases} u(x) = \arccos x & v'(x) = 1 \\ u'(x) = \dfrac{-1}{\sqrt{1-x^2}} & v(x) = x \end{cases}$

then

$$\begin{aligned} I &= x \arccos x - \int \frac{-x}{\sqrt{1-x^2}} dx \\ &= x \arccos x - \sqrt{1-x^2} + c. \end{aligned}$$

6. $I = \displaystyle\int x^2 \ln x dx$, *we set* $\begin{cases} u(x) = \ln x & v'(x) = x^2 \\ u'(x) = \dfrac{1}{x} & v(x) = \dfrac{x^3}{3} \end{cases}$

thus

$$I = \frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + c.$$

where c is a constant in $\mathbb{R}$.

4.5.2 Integration by Substitution

Computing integrals is not always straightforward. Sometimes, we perform the following change of variable in the integral $\displaystyle\int f(x)\, dx$:

$$x = \varphi(t),$$

where $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, and $\varphi : [\alpha, \beta] \longrightarrow [a, b]$ is a C^1 bijective function such that $\varphi(\alpha) = a$ and $\varphi(\beta) = b$. Then

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)\, dt.$$

Remark 4.5.1. *It is sometimes preferable to choose the change of variable in the form $t = \psi(x)$ instead of $x = \varphi(t)$.*

Examples 4.5.2. *Compute the following integrals:* $(c \in \mathbb{R})$

1. $I = \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$. *Let* $t = \cos x$, *then* $dt = -\sin x \, dx$, *hence*
$$I = \int \frac{-1}{t} \, dt = -\ln |t| + c = -\ln |\cos x| + c.$$

2. $I = \int \frac{(\ln x)^3}{x} \, dx$. *Let* $\ln x = t$, *then* $dt = \frac{dx}{x}$, *hence*
$$I = \int t^3 \, dt = \frac{t^4}{4} + c = \frac{(\ln x)^4}{4} + c.$$

3. $I = \int \frac{x}{1 + x^4} \, dx$. *Let* $x^2 = t$, *then* $dt = 2x \, dx$, *hence*
$$I = \int \frac{1}{2(1 + t^2)} \, dt = \frac{1}{2} \arctan t + c = \frac{1}{2} \arctan(x^2) + c.$$

4. $I = \int \sqrt{\sin x} \cos x \, dx$. *Let* $\sin x = t$, *then* $dt = \cos x \, dx$, *hence*
$$I = \int \sqrt{t} \, dt = \frac{2}{3} t^{3/2} + c = \frac{2}{3} (\sin x)^{3/2} + c.$$

5. $I = \int_0^{\pi} \cos^2 x \, \sin x \, dx$. *Let* $\cos x = t$, *then* $dt = -\sin x \, dx$, *and for* $x = 0 \Rightarrow t = 1$, $x = \pi \Rightarrow t = -1$, *hence*
$$I = \int_1^{-1} -t^2 \, dt = \int_{-1}^{1} t^2 \, dt = \left[\frac{t^3}{3} \right]_{-1}^{1} = \frac{2}{3}.$$

6. $I = \int \frac{x}{\sqrt{1 + x}} \, dx$. *Let* $\sqrt{1 + x} = t \Rightarrow x = t^2 - 1$, *then* $dx = 2t \, dt$, *hence*
$$I = \int \frac{2t(t^2 - 1)}{t} \, dt = \int 2(t^2 - 1) \, dt = \frac{2}{3} t^3 - 2t + c = \frac{2}{3} (1 + x)^{3/2} - 2\sqrt{1 + x} + c.$$

7. $I = \int \frac{dx}{x^2 + a^2} = \frac{1}{a^2} \int \frac{dx}{\frac{x^2}{a^2} + 1}$, *with* $a \neq 0$. *Let* $t = \frac{x}{a}$, *then* $dx = a \, dt$, *hence*
$$I = \frac{1}{a} \int \frac{dt}{t^2 + 1} = \frac{1}{a} \arctan\left(\frac{x}{a} \right) + c.$$

8. $I = \displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}$*, with* $a > 0$*. Let* $t = \frac{x}{a}$*, then*

$$I = \arcsin\left(\frac{x}{a}\right) + c.$$

9. $I = \displaystyle\int_0^1 \sqrt{1 - x^2}\, dx$*. Let* $x = \sin t$*, then* $dx = \cos t\, dt$*, and for* $x = 0 \Rightarrow t = 0$*,* $x = 1 \Rightarrow t = \pi/2$*, hence*

$$I = \int_0^{\pi/2} \cos^2 t\, dt = \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2t)\, dt = \frac{\pi}{4}.$$

10. $I = \displaystyle\int \frac{dx}{\tan^3 x} = \int \frac{\cos^3 x}{\sin^3 x}\, dx = \int \frac{(1 - \sin^2 x)\cos x}{\sin^3 x}\, dx$*, hence*

$$I = \int \cos x \, \sin^{-3} x\, dx - \int \frac{\cos x}{\sin x}\, dx = -\frac{1}{2\sin^2 x} - \ln|\sin x| + c.$$

4.5.3 Integration of Rational Functions

Definition 4.5.1. *Let* P *and* Q *be real polynomials, with* $Q \neq 0$*. The function*

$$x \longmapsto \frac{P(x)}{Q(x)}$$

is called a rational function*. If*

$$Q(x) = (x - x_0)^m Q_1(x) \quad \text{with } Q_1(x_0) \neq 0,$$

then x_0 *is called a* pole of order m *of the fraction* $\dfrac{P(x)}{Q(x)}$*. A rational function is called* proper *if* $\deg P < \deg Q$*.*

Definition 4.5.2. *The functions*

$$x \longmapsto \frac{A}{(x - x_0)^n}, \quad x \longmapsto \frac{px + q}{(ax^2 + bx + c)^m},$$

where $n, m \in \mathbb{N}$ *and* $A, a, b, c, p, q \in \mathbb{R}$ *with* $\Delta = b^2 - 4ac < 0$*, are called* simple elements of the first and second kind*, respectively.*

Theorem 4.5.2.

1. *Every proper rational function can be decomposed into a sum of simple elements.*

2. *If* $\deg P \geq \deg Q$*, perform polynomial division first to write*

$$\frac{P(x)}{Q(x)} = A(x) + \frac{R(x)}{Q(x)}, \quad \deg R < \deg Q,$$

then decompose the proper fraction $\dfrac{R(x)}{Q(x)}$ *into simple elements.*

Example 4.5.1. *If* $\deg P < 7$ *and* $p^2 - 4q < 0$*, then the rational function*

$$f(x) = \frac{P(x)}{(x - x_0)^3 (x^2 + px + q)^2}$$

can be decomposed in the following form:

$$f(x) = \frac{a_1}{x - x_0} + \frac{a_2}{(x - x_0)^2} + \frac{a_3}{(x - x_0)^3} + \frac{b_1 x + c_1}{x^2 + px + q} + \frac{b_2 x + c_2}{(x^2 + px + q)^2}.$$

Exercise 4.5.1.
Decompose the following rational functions into a sum of simple elements:

$$1) f(x) = \frac{2x - 1}{(x - 2)(x + 3)}, \qquad 2) f(x) = \frac{2x - 1}{(x - 2)^3}, \qquad 3) f(x) = \frac{3x^5 + 2x^4 + x^2 + 3x + 2}{x^2 + 1},$$

$$4) f(x) = \frac{x^3}{(x + 1)(x^2 + 4)^2}, \qquad 5) f(x) = \frac{1}{(x^4 - 1)^2}.$$

Solution 4.5.1.

$$1) f(x) = \frac{3}{5(x - 2)} + \frac{7}{5(x + 3)}, \qquad 2) f(x) = \frac{0}{(x - 2)} + \frac{2}{(x - 2)^2} + \frac{3}{(x - 2)^3},$$

$$3) f(x) = 3x^3 + 2x^2 - 3x - 1 + \frac{3(2x + 1)}{x^2 + 1}, \qquad 4) f(x) = \frac{1}{25(x + 1)} + \frac{x + 24}{25(x^2 + 4)} - \frac{4x + 16}{5(x^2 + 4)^2}$$

$$5) f(x) = \frac{a}{x - 1} + \frac{b}{(x - 1)^2} + \frac{c}{x + 1} + \frac{d}{(x + 1)^2} + \frac{ex + f}{1 + x^2} + \frac{gx + h}{(1 + x^2)^2}.$$

Example 4.5.2. *Compute the following integrals:*

$$1)I = \int \frac{1}{x(1+x^2)}dx, \qquad 2)I = \int \frac{1}{x(1+x)(1+x+x^2)}dx,$$

$$3)I = \int \frac{2x^2}{x-1}dx, \qquad 4)I = \int \frac{x^3}{x^2-1}dx.$$

Solution 4.5.2.

$1)I = \displaystyle\int \frac{1}{x(1+x^2)}dx$. *We begin by decomposing the integrand into partial fractions:*

$$\frac{1}{x(1+x^2)} = \frac{a}{x} + \frac{bx+c}{1+x^2} = \frac{x^2(a+b)+cx+a}{x(1+x^2)},$$

By comparing coefficients, we find $a = 1$, $b = -1$, and $c = 0$. Hence,

$$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2},$$

Integrating both terms gives

$$I = \int \frac{1}{x}dx - \int \frac{x}{1+x^2}dx = \ln|x| - \frac{1}{2}\ln(1+x^2) + c.$$

$2)I = \displaystyle\int \frac{1}{x(1+x)(1+x+x^2)}dx$. *We decompose the rational function as follows:*

$$\frac{1}{x(1+x)(1+x+x^2)} = \frac{1}{x} - \frac{1}{1+x} - \frac{1}{1+x+x^2}.$$

Therefore,

$$I = \int \frac{1}{x}dx - \int \frac{1}{1+x}dx - \int \frac{1}{1+x+x^2}dx = \ln|x| - \ln|1+x| - \frac{2}{\sqrt{3}}\arctan\left[\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right] + c,$$

since

$$\begin{aligned}\int \frac{1}{x^2+x+1}dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx &= \frac{4}{3}\int \frac{1}{\left[\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right]^2+1}dx \\ &= \frac{2}{\sqrt{3}}\arctan\left[\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right] + c.\end{aligned}$$

3)$I = \int \frac{2x^2}{x-1} dx$. *Performing the polynomial division, we obtain*

$$\frac{2x^2}{x-1} = 2x + 2 + \frac{2}{x-1}.$$

Hence,

$$I = \int (2x+2)\,dx + \int \frac{2}{x-1}\,dx = x^2 + 2x + 2\ln|x-1| + C.$$

4)$I = \int \frac{x^3}{x^2-1}\,dx$. *We rewrite the numerator as follows:*

$$\begin{aligned} I &= \int \frac{x^3 - x + x}{x^2-1}\,dx = \int \frac{x^3-x}{x^2-1}\,dx + \int \frac{x}{x^2-1}\,dx \\ &= \int x\,dx + \frac{1}{2}\int \frac{2x}{x^2-1}\,dx = \frac{x^2}{2} + \frac{1}{2}\ln|x^2-1| + C, \end{aligned}$$

where $C \in \mathbb{R}$.

4.5.4 Integration of the form $\int R(e^x)\,dx$

Let R be a rational function. Set $t = e^x$; then $dt = e^x dx$, so $dx = \frac{dt}{t}$.

Example 4.5.3.

$$\int \frac{dx}{1+e^x} = \int \frac{dt}{t(1+t)} = \int \frac{dt}{t} - \int \frac{dt}{1+t} = \ln\left(\frac{t}{1+t}\right) + c = \ln\left(\frac{e^x}{1+e^x}\right) + c.$$

4.5.5 Integration of the forms $\int \frac{\alpha x+\beta}{ax^2+bx+c}\,dx$ and $\int \frac{\alpha x+\beta}{\sqrt{ax^2+bx+c}}\,dx$

Compute the following integrals:

1. $\displaystyle\int \frac{dx}{2x^2-2x+1} = arctg(2x-1) + c.$

2. $\displaystyle\int \frac{2dx}{x^2-6x+5} = \frac{1}{2}\ln|\frac{x-5}{x-1}| + c.$

3. $\displaystyle\int \frac{(6x-7)dx}{3x^2-7x+11} = \ln|3x^2-7x+11| + c.$

4. $\displaystyle\int \frac{dx}{\sqrt{x^2+x+1}} = arg\sinh(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}) + c = \ln(x + \frac{1}{2} + \sqrt{x^2+x+1}) + c'.$

5. $\displaystyle\int \frac{dx}{\sqrt{5-6x-3x^2}} = \frac{1}{\sqrt{3}} arc\sin\frac{\sqrt{3}(x+1)}{\sqrt{8}} + c.$

6. $$\begin{aligned}\int \frac{(3x-2)dx}{5x^2-3x+2} &= \int \frac{\frac{3}{10}(10x-3)+\frac{9}{10}-2}{5x^2-3x+2} \\ &= \frac{3}{10}\ln|5x^2-3x+2| - \frac{11}{5\sqrt{31}} arctg\frac{10x-3}{\sqrt{31}} + c.\end{aligned}$$

7. $$\begin{aligned}\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx &= \int \frac{\frac{5}{2}(2x+4)-5\frac{4}{2}+3}{\sqrt{x^2+4x+10}}dx \\ &= 5\sqrt{x^2+4x+10} - 7\ln|x+2+\sqrt{x^2+4x+10}| + c \\ &= 5\sqrt{x^2+4x+10} - 7arg\sinh(\frac{x+2}{\sqrt{6}}) + c'.\end{aligned}$$

4.5.6 Integration of the form $\int R(\sin x, \cos x)\, dx$

We make the substitution

$$\tan\left(\frac{x}{2}\right) = t, \quad \text{so that} \quad dx = \frac{2\,dt}{1+t^2},$$

and the trigonometric functions $\sin x$ and $\cos x$ can be expressed as:

$$\sin x = \frac{2\sin(\frac{x}{2})\cos(\frac{x}{2})}{\cos^2(\frac{x}{2})+\sin^2(\frac{x}{2})} = \frac{2t}{1+t^2}; \quad \cos x = \frac{\cos^2(\frac{x}{2})-\sin^2(\frac{x}{2})}{\cos^2(\frac{x}{2})+\sin^2(\frac{x}{2})} = \frac{1-t^2}{1+t^2}.$$

Examples 4.5.3.

$$1) \int \frac{1}{\sin x}dx = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int \frac{dt}{t} = \ln|tg(\frac{x}{2})| + c.$$

$$2) \int \frac{dx}{1+\sin x+\cos x} = \int \frac{2dt}{(1+t^2)(1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2})} = \int \frac{dt}{1+t} = \ln|1+tg(\frac{x}{2})| + c.$$

Special cases:

1. If the integral can be written in the form $I = \displaystyle\int R(\sin x)\,\cos x\,dx$, try the substitution $\sin x = t$.
2. If the integral can be written in the form $I = \displaystyle\int R(\cos x)\,\sin x\,dx$, try the substitution $\cos x = t$.
3. If the integral can be written in the form $I = \displaystyle\int R(\tan x)\,\frac{dx}{\cos^2 x}$, perform the substitution $\tan x = t$.

Example 4.5.4. *Compute the following integrals:*

$$\int \cos^4 x \sin^3 x dx; \quad \int \frac{\sin^5 x}{\cos x} dx; \quad \int \frac{\cos x dx}{3 - \cos^2 x}; \quad \int tg^3 x dx; \quad \int \frac{1 + tg^2 x}{1 + 2tg^2 x} dx.$$

4.5.7 Integration of the form $\int R(\sinh x, \cosh x)\,dx$

The same principle used for trigonometric functions can be applied here. In the general case, we set

$$\tanh\left(\frac{x}{2}\right) = t, \quad \text{so that} \quad dx = \frac{2\,dt}{1 - t^2}, \quad \sinh x = \frac{2t}{1 - t^2}, \quad \cosh x = \frac{1 + t^2}{1 - t^2}.$$

1. If the integral is of the form $I = \displaystyle\int R(\sinh x)\,\cosh x\,dx$, use the substitution $\sinh x = t$.
2. If the integral is of the form $I = \displaystyle\int R(\cosh x)\,\sinh x\,dx$, use the substitution $\cosh x = t$.

Example 4.5.5.

$$\int \frac{dx}{shx} = \int \frac{2dt}{2t} = \ln|\tanh(\frac{x}{2})| + c.$$

Remark 4.5.2. *Integrals of the form*

$$J = \int R(\sinh x, \cosh x)\, dx$$

can also be evaluated by using the substitution $e^x = t$.

4.5.8 Integrals of the form $\int \sqrt{ax^2+bx+c}\, dx$

For integrals of this type one typically uses trigonometric or hyperbolic substitutions.

1. $\int f\big(x, \sqrt{a^2-x^2}\big)\, dx$. Use the substitution $x = a\sin t$.
   Alternatively, by integration by parts one gets
   $$\int \sqrt{a^2-x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin(\frac{x}{a}) + c.$$

2. $\int f\big(x, \sqrt{a^2+x^2}\big)\, dx$. Use the substitution $x = a\sinh t$.
   Alternatively, by integration by parts,
   $$\int \sqrt{a^2+x^2}dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}argsh(\frac{x}{a}) + c.$$

3. $\int f\big(x, \sqrt{x^2-a^2}\big)\, dx$. For $x > a > 0$ use $x = a\cosh t$ (for $x < -a < 0$ use $x = -a\cosh t$, since $\underline{\cosh t \geq 1}$). Alternatively, by integration by parts,
   $$\int \sqrt{x^2-a^2}dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}argch(\frac{x}{a}) + c.$$

Example 4.5.6. *Compute the following integrals:*

$$\int \sqrt{1+x^2}dx; \quad \int \frac{x+1}{\sqrt{1-x^2}}dx; \quad \int \sqrt{3-2x-x^2}dx.$$

Exercise 4.5.2. *Calculate* $\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1}dx$ *(p403 Piskounov).*

Chapter 5

Taylor Expansions

5.1 Taylor Formulas

Let f be a continuous function on a closed interval $[a, b]$, differentiable at some point $x_0 \in]a, b[$. Then, near x_0, the function can be written as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \underbrace{(x - x_0)\varepsilon(x)}_{R(x)}, \quad \text{where } \lim_{x \to x_0} \varepsilon(x) = 0.$$

This means that f can be approximated by the first-degree polynomial

$$p(x) = f(x_0) + f'(x_0)(x - x_0).$$

Here, $R(x)$ represents the error of this approximation, which tends to zero as $x \to x_0$.

The Taylor formula generalizes this result, showing that if f is sufficiently smooth ($f \in C^n(\mathbb{R})$), then, in a neighborhood of x_0, it can be approximated by a polynomial P_n of degree n:

$$\begin{aligned} f(x) &= \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n}_{P_n(x)} \\ &+ \underbrace{R_n(x)}_{\text{remainder}}. \end{aligned}$$

The term $R_n(x)$ is called the "remainder of order n". Depending on the smoothness of f, several equivalent forms of this remainder can be given, leading to different versions of the Taylor formula.

Theorem 5.1.1 (Taylor's Theorem with Lagrange Remainder)**.**
Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in C^n([a, b])$ and $f^{(n+1)}$ exists on $]a, b[$. Then, there exists a point $c \in]a, b[$ such that

$$\begin{aligned} f(b) &= f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n \\ &+ \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}. \end{aligned}$$

This result can also be written, setting $h = b - a$, as

$$f(a+h) = f(a) + \frac{f'(a)}{1!}h + \frac{f''(a)}{2!}h^2 + \cdots + \frac{f^{(n)}(a)}{n!}h^n + \frac{f^{(n+1)}(a+\theta h)}{(n+1)!}h^{n+1},$$

where $\theta \in]0, 1[$.

Remark 5.1.1. *If we take $a = 0$ and let $x = a + h$, we obtain the Maclaurin formula of order n:*

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}.$$

Example 5.1.1.
Since $e^x \in C^\infty(\mathbb{R})$, we have, for some $\theta \in]0, 1[$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}e^{\theta x}}{(n+1)!}.$$

Theorem 5.1.2 (Taylor's Formula with the Little-o Remainder).
Let $f \in C^n(I)$, where I is an interval containing x_0. Then, for every $x \in I$,

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &+ o\big((x - x_0)^n\big), \end{aligned}$$

where

$$\lim_{x \to x_0} \frac{o\big((x - x_0)^n\big)}{(x - x_0)^n} = 0.$$

Alternative form:

$$o\big((x - x_0)^n\big) = (x - x_0)^n \, \varepsilon(x), \quad \text{with} \quad \lim_{x \to x_0} \varepsilon(x) = 0.$$

Theorem 5.1.3 (Maclaurin's Formula with the Little-o Remainder).
Let $f \in C^n(I)$, where I is an interval containing 0. Then, for every $x \in I$,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + x^n \varepsilon(x),$$

where $\lim_{x \to 0} \varepsilon(x) = 0$.

Examples 5.1.1.

1. *For $f(x) = e^x$, since $f^{(n)}(x) = e^x$, then*

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + x^n \varepsilon(x).$$

2. *For $f(x) = \sin x$, since $f^{(n)}(x) = \sin\Big(x + n\frac{\pi}{2}\Big)$, then*

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + x^{2n+1} \varepsilon(x),$$

which is the Maclaurin expansion of order $2n + 1$.

3. For $f(x) = \cos x$*, since* $f^{(n)}(x) = \cos\Big(x + n\dfrac{\pi}{2}\Big)$,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + x^{2n}\varepsilon(x),$$

which is the Maclaurin expansion of order $2n$.

4. For $f(x) = (1+x)^\alpha$*, where* $\alpha \in \mathbb{R}$*,*
since $f^{(n)}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$*, then*

$$\begin{aligned}(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}x^n \\ &+ x^n\varepsilon(x).\end{aligned}$$

When $\alpha = n \in \mathbb{N}^*$*, we recover the binomial formula:*

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots 2\cdot 1}{n!}x^n.$$

For $\alpha = -1$*,*

$$\begin{aligned}(1+x)^{-1} &= \frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + x^n\varepsilon(x),\\ (1-x)^{-1} &= \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + x^n\varepsilon(x).\end{aligned}$$

where $\lim\limits_{x\to 0}\varepsilon(x) = 0$.

5.2 Taylor Expansions at the Origin

Definition 5.2.1. *Let* f *be a function defined in a neighborhood* V *of* 0*. We say that* f *admits a* ***Taylor expansion of order*** n *in this neighborhood if there exists a polynomial* P_n *of degree* n *such that*

$$\forall x \in V: \quad f(x) = P_n(x) + x^n\varepsilon(x) \quad \textit{where} \quad \lim_{x\to 0}\varepsilon(x) = 0,$$

or equivalently,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + x^n \varepsilon(x).$$

The polynomial P_n is called the ***Taylor polynomial of degree n*** *or* ***principal part of the expansion****.*

Theorem 5.2.1. *If f admits a Taylor expansion of order n in a neighborhood of 0, then this expansion is unique.*

Example 5.2.1.
(1) The function $f(x) = \dfrac{1}{1-x}$, defined for $x \neq 1$, admits a Taylor expansion of order n near 0, obtained by performing the division in increasing powers of x:

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots + \frac{x^{n+1}}{1-x} \\ &= 1 + x + x^2 + x^3 + \cdots + x^n \frac{x}{1-x}, \end{aligned}$$

with $\varepsilon(x) = \dfrac{x}{1-x} \longrightarrow 0$ as $x \to 0$.

(2) If Q is a polynomial, then Q admits a Taylor expansion near 0, and its expansion is itself, with $\varepsilon(x) = 0$.

Remark 5.2.1.

1. *The principal part of the Taylor expansion of an even function is even.*

2. *The principal part of the Taylor expansion of an odd function is odd.*

5.3 Obtaining Taylor Expansions by the Maclaurin-Young Formula

Theorem 5.3.1. *If $f^{(n)}(0)$ exists, then f admits a unique Taylor expansion of order n in a neighborhood of 0:*

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + x^n\epsilon(x),$$

where $\lim\limits_{x\to 0}\epsilon(x) = 0$.

<u>**Conclusion:**</u> If $f^{(n)}(0)$ exists, then f has a unique Taylor expansion of order n near 0; that is,

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + x^n\epsilon(x),$$

and by matching coefficients, we obtain

$$f(0) = a_0, \qquad \frac{f'(0)}{1!} = a_1, \quad \ldots, \quad \frac{f^{(n)}(0)}{n!} = a_n.$$

Remark 5.3.1. *If f admits a Maclaurin expansion, then it also admits a Taylor expansion near 0.*
However, a function may have a Taylor expansion without being differentiable. For example, the function

$$f(x) = x^{n+1}\sin\left(\frac{1}{x}\right)$$

admits a Taylor expansion since

$$f(x) = 0 + x^n\big(x\sin(\frac{1}{x})\big), \quad \textit{and} \quad \lim_{x\to 0}\varepsilon(x) = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0,$$

but it does not admit a Maclaurin expansion because it is not defined at $x = 0$.

5.4 Operations on Taylor Expansions

Theorem 5.4.1. *Let f and g be functions defined in a neighborhood of* 0*, each having a Taylor expansion of order n about* 0*. Then:*

1. *The sum $f+g$ also has a Taylor expansion of order n, whose polynomial part is the sum of the polynomial parts of f and g.*

2. *The product fg has a Taylor expansion of order n, whose polynomial part is obtained by multiplying the polynomial parts of f and g and keeping only the terms of degree less than or equal to n.*

3. *The quotient $\dfrac{f}{g}$ has a Taylor expansion of order n (provided that $g(0) \neq 0$), whose polynomial part is obtained by dividing the polynomial part of f by that of g, keeping terms up to degree n.*

Examples 5.4.1.

1. *Find the expansion of* $\cosh x$ *and* $\sinh x$ *about* 0.

   Since
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + o(x^n),$$
   and
$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n x^n}{n!} + o(x^n),$$
   such that $\lim\limits_{x \longrightarrow 0} \dfrac{o(x^n)}{x^n} = 0$. *We get*
$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + o(x^{2n}) \ / \ \lim_{x \longrightarrow 0} \frac{o(x^{2n})}{x^{2n}} = 0,$$
$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1}) \ / \ \lim_{x \longrightarrow 0} \frac{o(x^{2n+1})}{x^{2n+1}} = 0.$$

2. Find the fifth-degree expansion of $\cos^2 x$ *about* 0*:*

$$\cos^2 x = \big(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)\big)^2 = 1 - x^2 + \frac{x^4}{3} + o(x^5),$$

such that $\lim\limits_{x \longrightarrow 0} \dfrac{o(x^5)}{x^5} = 0.$

3. Find the fifth-degree expansion of $\tan x = \dfrac{\sin x}{\cos x}$ *about* 0*:*

$$\tan x = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)}{1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)} = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5),$$

such that $\lim\limits_{x \longrightarrow 0} \dfrac{o(x^5)}{x^5} = 0.$

4. Find the fifth-degree expansion of $\tanh x = \dfrac{\sinh x}{\cosh x}$ *about* 0*:*

$$\tanh x = \frac{x + \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)}{1 + \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)} = x - \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5),$$

such that $\lim\limits_{x \longrightarrow 0} \dfrac{o(x^5)}{x^5} = 0.$

Theorem 5.4.2 (Taylor expansion of a composite function)**.**
Let f *and* u *be two functions that each admit a Taylor expansion of order* n *near* 0*, such that*

$$f(x) = P(x) + o(x^n) \quad \textit{and} \quad u(x) = Q(x) + o(x^n),$$

where $\lim\limits_{x \to 0} \dfrac{o(x^n)}{x^n} = 0$ *and* $u(0) = 0$*. Then the composite function* $f \circ u$ *admits a Taylor expansion of order* n *near* 0*, and its principal part is obtained by computing* $(P \circ Q)(x)$ *and keeping only the terms of degree less than or equal to* n*.*

Example 5.4.1. *Find the fifth-order Taylor expansion at 0 of $e^{\sin x}$, $e^{\cos x}$ and $\sqrt{\cos x}$.*

1.

$$e^{\sin x} = e^{x-\frac{x^3}{6}+\frac{x^5}{120}+o(x^5)},$$

we set $u = x - \dfrac{x^3}{6} + \dfrac{x^5}{120} + o(x^5)$ *so that* $u(0) = 0$*, then*

$$\begin{aligned}
e^{\sin x} = e^u &= 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + o(u^5) \\
&= 1 + (x - \frac{x^3}{6} + \frac{x^5}{120}) + \frac{(x - \frac{x^3}{6} + \frac{x^5}{120})^2}{2!} + \frac{(x - \frac{x^3}{6} + \frac{x^5}{120})^3}{3!} \\
&\quad + \frac{(x - \frac{x^3}{6} + \frac{x^5}{120})^4}{4!} + \frac{(x - \frac{x^3}{6} + \frac{x^5}{120})^5}{5!} + o(x^5) \\
&= 1 + x - \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^2}{2} - \frac{x^4}{6} + \frac{x^3}{6} - \frac{x^5}{12} + \frac{x^4}{24} + \frac{x^5}{120} + o(x^5) \\
&= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + o(x^5),
\end{aligned}$$

such that $\lim\limits_{x \longrightarrow 0} \dfrac{o(x^5)}{x^5} = 0.$

2.

$$e^{\cos x} = e^{1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^5)} = e.e^{-\frac{x^2}{2}+\frac{x^4}{24}+o(x^5)},$$

we set $u = -\dfrac{x^2}{2} + \dfrac{x^4}{24} + o(x^5)$ *so that* $u(0) = 0$*, then*

$$\begin{aligned}
e^{\cos x} = e.e^u &= e[1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + o(u^5)] \\
&= e[1 + (-\frac{x^2}{2} + \frac{x^4}{24}) + \frac{(-\frac{x^2}{2} + \frac{x^4}{24})^2}{2!} + o(x^5)] \\
&= e[1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^4}{8} + o(x^5)] \\
&= e[1 - \frac{x^2}{2} + \frac{x^4}{6} + o(x^5)],
\end{aligned}$$

such that $\lim_{x \longrightarrow 0} \frac{o(x^5)}{x^5} = 0.$

3.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5),$$

we set $u = -\frac{x^2}{2} + \frac{x^4}{24} + o(x^5)$ *so that* $u(0) = 0$*, then*

$$\begin{aligned}\sqrt{\cos x} &= \sqrt{1+u} = 1 + \frac{u}{2} - \frac{u^2}{8} + o(u^2) \\ &= 1 + \frac{1}{2}(-\frac{x^2}{2} + \frac{x^4}{24}) - \frac{1}{8}(-\frac{x^2}{2} + \frac{x^4}{24})^2 + o(x^5) \\ &= 1 - \frac{x^2}{4} - \frac{x^4}{96} + o(x^5),\end{aligned}$$

such that $\lim_{x \longrightarrow 0} \frac{o(x^5)}{x^5} = 0.$

5.5 Differentiation and Integration of Taylor Expansions

Propertys 5.5.1. *Let* $f : I \to \mathbb{R}$ *be a function defined on an interval* I *containing* 0*, and suppose that* f *admits a Taylor expansion of order* n *about* 0*. Then:*

1. *If* f *is differentiable on* I*, its derivative* f' *also admits a Taylor expansion of order* $n-1$ *about* 0*, whose principal part is obtained by differentiating the principal part of* f*; that is,*

$$f'(x) = p'(x) + o(x^{n-1}), \qquad \lim_{x \to 0} \frac{o(x^{n-1})}{x^{n-1}} = 0.$$

2. *If* f *admits an antiderivative* F*, then* F *admits a Taylor expansion of order* $n+1$ *about* 0*, whose principal part is obtained by integrating the*

principal part of f and adding the constant $F(0)$; that is,

$$F(x) = F(0) + \int p(x)\,dx + o(x^{n+1}), \qquad \lim_{x\to 0} \frac{o(x^{n+1})}{x^{n+1}} = 0.$$

Examples 5.5.1.

1. *From the expansion of $\dfrac{1}{1-x}$, we can obtain that of $\dfrac{1}{(1-x)^2}$ by differentiation:*

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + o(x^n), \qquad \lim_{x\to 0} \frac{o(x^n)}{x^n} = 0,$$

hence

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + o(x^{n-1}),$$

with $\lim_{x\to 0} \dfrac{o(x^{n-1})}{x^{n-1}} = 0.$

2. *Integration example:*

$$(\ln(1+x))' = \frac{1}{1+x} = 1-x+x^2-x^3+\cdots+(-1)^n x^n + o(x^n), \quad \lim_{x\to 0} \frac{o(x^n)}{x^n} = 0,$$

therefore

$$\begin{aligned} \ln(1+x) &= \ln(1+0) + \int (1 - x + x^2 - x^3 + \cdots + (-1)^n x^n)\,dx + o(x^{n+1}) \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + o(x^{n+1}), \end{aligned}$$

with $\lim_{x\to 0} \dfrac{o(x^{n+1})}{x^{n+1}} = 0.$

By replacing x by $-x$, we get

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots - \frac{x^{n+1}}{n+1} + o(x^{n+1}).$$

3. Find the Taylor expansion of order $2n+1$ *about* 0 *for* $\arctan x$ *and* $\arcsin x$*:*

$$(\arctan x)' = \frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} + o(x^{2n}),$$

hence

$$\begin{aligned}\arctan x &= \arctan 0 + \int (1 - x^2 + x^4 - \cdots + (-1)^n x^{2n})\,dx + o(x^{2n+1}) \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+1}),\end{aligned}$$

with $\lim_{x\to 0} \frac{o(x^{2n+1})}{x^{2n+1}} = 0.$

Likewise,

$$\begin{aligned}(\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} \\ &= 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \cdots + \frac{3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^{2n} + o(x^{2n}),\end{aligned}$$

therefore

$$\begin{aligned}\arcsin x = \arcsin 0 + \int \left(1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \cdots + \frac{3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^{2n}\right) dx \\ + o(x^{2n+1}) \\ = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots + \frac{3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\frac{x^{2n+1}}{2n+1} + o(x^{2n+1}),\end{aligned}$$

with $\lim_{x\to 0} \frac{o(x^{2n+1})}{x^{2n+1}} = 0.$

5.6 Taylor expansions near $x_0 \neq 0$ and at infinity

Definition 5.6.1.

1. *Let $f : I \to \mathbb{R}$ with I an interval and $x_0 \in I$. We say that f admits a* Taylor expansion of order n at x_0 *if one can write*

$$f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + o\big((x - x_0)^n\big),$$

such that $\displaystyle\lim_{x\to x_0} \frac{o\big((x - x_0)^n\big)}{(x - x_0)^n} = 0.$

2. *We say that f admits an* asymptotic expansion of order n at infinity *if one can write*

$$f(x) = c_0 + \frac{c_1}{x} + \cdots + \frac{c_n}{x^n} + o\Big(\frac{1}{x^n}\Big), \quad \lim_{x\to\infty} \frac{o\big(\frac{1}{x^n}\big)}{\frac{1}{x^n}} = 0.$$

Remark 5.6.1. *By the change of variable $y = x - x_0$ (so $y \to 0$ as $x \to x_0$) one reduces a Taylor expansion at x_0 to one at 0. Similarly, the change of variable $y = 1/x$ (so $y \to 0$ as $x \to \infty$) reduces an expansion at infinity to one at 0.*

Example 5.6.1.

1. ***Expansion of order n for e^x about $x_0 = 1$.***
Put $y = x - 1$ (so y is near 0). Then

$$e^x = e^{y+1} = e\,e^y = e\Big(1 + y + \frac{y^2}{2!} + \cdots + \frac{y^n}{n!} + o(y^n)\Big),$$

hence

$$e^x = e\Big(1 + (x - 1) + \frac{(x - 1)^2}{2!} + \cdots + \frac{(x - 1)^n}{n!} + o\big((x - 1)^n\big)\Big).$$

2. ***Expansion of order*** $2n$ ***for*** $\sin x$ ***about*** $x_0 = \frac{\pi}{2}$**.**

Put $y = x - \frac{\pi}{2}$ *(so* y *is near* 0*). Using* $\sin(x) = \sin(y + \frac{\pi}{2}) = \cos y$ *and the cosine expansion,*

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots + \frac{(-1)^n y^{2n}}{(2n)!} + o(y^{2n}),$$

we obtain

$$\sin x = 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} + \cdots + \frac{(-1)^n (x-\frac{\pi}{2})^{2n}}{(2n)!} + o\big((x-\frac{\pi}{2})^{2n}\big).$$

3. ***Expansion of*** $e^{\sqrt{x}}$ ***about*** $x_0 = 1$ ***to order*** 3**.**

Put $y = \sqrt{x} - 1$ *or equivalently* $x = (1+y)^2$ *with* $y \to 0$*. One may expand* $e^{\sqrt{x}} = e^{1+y} = e(1 + y + \frac{y^2}{2} + \frac{y^3}{6} + o(y^3))$ *and then substitute* $y = \sqrt{x} - 1$ *and re-expand in powers of* $x - 1$ *up to order* 3*.*

4. ***Expansion of*** $\ln(1+3x)$ ***about*** $x_0 = 1$ ***to order*** 3**.**

Put $y = x - 1$*. Then*

$$\ln(1+3x) = \ln(4+3y) = \ln\big(4(1+\frac{3}{4}y)\big) = \ln 4 + \ln\big(1 + \frac{3}{4}y\big).$$

Expanding $\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3)$ *with* $t = \frac{3}{4}y$ *yields*

$$\ln(1+3x) = \ln 4 + \frac{3}{4}y - \frac{1}{2}\Big(\frac{3}{4}y\Big)^2 + \frac{1}{3}\Big(\frac{3}{4}y\Big)^3 + o(y^3),$$

i.e.

$$\ln(1+3x) = \ln 4 + \frac{3}{4}(x-1) - \frac{1}{2}\Big(\frac{3(x-1)}{4}\Big)^2 + \frac{1}{3}\Big(\frac{3(x-1)}{4}\Big)^3 + o\big((x-1)^3\big).$$

5. ***Expansion at*** $+\infty$ ***of*** $f(x) = \dfrac{3x^2+x+1}{2x^2+5x+1}$ ***to order*** 2**.**

Put $y = \frac{1}{x}$ *(so* $y \to 0$ *as* $x \to \infty$*). Then*

$$f\Big(\frac{1}{y}\Big) = \frac{\frac{3}{y^2} + \frac{1}{y} + 1}{\frac{2}{y^2} + \frac{5}{y} + 1} = \frac{3 + y + y^2}{2 + 5y + y^2}.$$

Expanding in powers of y gives

$$f\Big(\frac{1}{y}\Big) = \frac{3}{2} - \frac{13}{4}y + \frac{63}{8}y^2 + o(y^2),$$

hence, returning to x,

$$f(x) = \frac{3}{2} - \frac{13}{4x} + \frac{63}{8x^2} + o\Big(\frac{1}{x^2}\Big).$$

6. ***Expansion at** $+\infty$ **of** $f(x) = \dfrac{x}{x-1}$ **to order** 3**, and of** $\ln\Big(\dfrac{2x+1}{x}\Big)$**.***
   For $f(x) = \dfrac{x}{x-1}$, expand in powers of $y = 1/x$:

   $$\frac{x}{x-1} = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + o\Big(\frac{1}{x^3}\Big).$$

   For the logarithm,

   $$\ln\Big(\frac{2x+1}{x}\Big) = \ln\Big(2 + \frac{1}{x}\Big) = \ln 2 + \ln\Big(1 + \frac{1}{2x}\Big),$$

   hence the expansion at infinity is

   $$\ln\Big(\frac{2x+1}{x}\Big) = \ln 2 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{24x^3} + o\Big(\frac{1}{x^3}\Big).$$

Remark 5.6.2.

1. *If f admits a Taylor expansion of order n at x_0, that is,*

   $$f(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n + o\big((x - x_0)^n\big),$$

   then

   $$\lim_{x\to x_0} f(x) = a_0.$$

   By contraposition, if $\lim\limits_{x\to x_0} f(x)$ does not exist or equals ∞, then f does not admit a Taylor expansion of order n at x_0.

For example:

$$f(x) = \ln|x| \xrightarrow[x\longrightarrow 0]{} -\infty \quad \Longrightarrow \quad f \textit{ has no Taylor expansion at } 0,$$

and

$f(x) = \sin\left(\dfrac{1}{x}\right)$ *has no limit as* $x \longrightarrow 0$, *hence no Taylor expansion at* 0.

$$2.\ \begin{cases} \textit{If } f \textit{ is continuous at } x_0 \\ \textit{and } f \textit{ admits a first-order} \\ \textit{Taylor expansion at } x_0. \end{cases} \Longrightarrow \begin{cases} \lim\limits_{x\longrightarrow x_0} f(x) = a_0 = f(x_0) \textit{ and} \\ f \textit{ is differentiable at } x_0 \textit{ and} \\ f'(x_0) = a_1 \end{cases}$$

Indeed,

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \big[a_1 + a_2(x-x_0) + \cdots + a_n(x-x_0)^{n-1} + o\big((x-x_0)^{n-1}\big)\big] = a_1.$$

If f *is continuous but not differentiable at* x_0, *then* f *does not admit a first-order Taylor expansion there.*

For instance, $f(x) = \sqrt[3]{x-1}$ *is continuous at* $x=1$ *but not differentiable since*

$$\lim_{x\to 1} \frac{\sqrt[3]{x-1}-0}{x-1} = \lim_{x\to 1} \frac{1}{(x-1)^{2/3}} = +\infty,$$

hence f *admits no Taylor expansion at* $x=1$.

Definition 5.6.2 (Generalized Expansion)**.**
Let f *be a function that does not admit a Taylor expansion at* x_0, *but such that the function*

$$x \longmapsto (x-x_0)^\alpha f(x), \qquad \alpha \in \mathbb{Q}_+^*,$$

does admit one. Then

$$(x-x_0)^\alpha f(x) = a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n + o\big((x-x_0)^n\big),$$

and consequently,

$$f(x) = \frac{1}{(x-x_0)^\alpha}\Big[a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n + o\big((x-x_0)^n\big)\Big].$$

This expression is called the generalized expansion *of f.*

Example 5.6.2. *Find the generalized expansion of order 3 at 0 of*

$$f(x) = \cot x = \frac{\cos x}{\sin x}.$$

Since $\lim\limits_{x\to 0} \cot x = +\infty$, f does not admit a Taylor expansion at 0.

$$\begin{aligned}
\cot x = \frac{\cos x}{\sin x} &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)}{x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)} \\
&= \frac{1}{x} \cdot \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)}{1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^4)} \\
&= \frac{1}{x}\Big(1 - \frac{x^2}{3} - \frac{x^4}{45} + o(x^4)\Big) \\
&= \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} + o(x^3).
\end{aligned}$$

Compute likewise the generalized expansion of

$$f(x) = \frac{1}{x - x^2}$$

at $x = 0$.

5.7 Applications of Taylor Expansions

Taylor expansions and generalized expansions are particularly useful for computing limits, approximating functions near a point, and determining the asymptotic behavior of functions at infinity, including the identification of asymptotes.

1) **Computing limits.** Example:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x + o(x)}{x} = 1.$$

2) **Finding asymptotes using generalized expansions.**

If $\lim_{x\to\infty} f(x) = \infty$, then f may have an asymptote near infinity. To find the equation of this line, we compute the generalized expansion of order 1 of f at infinity.

Let $y = \dfrac{1}{x}$ in order to move to a neighborhood of 0. If

$$f\Big(\frac{1}{y}\Big) = \frac{1}{y}\big(a_0 + a_1 y + a_2 y^2 + o(y^2)\big) = \frac{a_0}{y} + a_1 + a_2 y + o(y),$$

then returning to the variable x gives

$$f(x) = a_0 x + a_1 + \frac{a_2}{x} + o\Big(\frac{1}{x}\Big).$$

Consequently,

$$\lim_{x\to\infty} \big(f(x) - (a_0 x + a_1)\big) = \lim_{x\to\infty} \Big(\frac{a_2}{x} + o\Big(\frac{1}{x}\Big)\Big) = 0,$$

so the line $y = a_0 x + a_1$ is an asymptote of the graph of f at $+\infty$.

To determine the position of the graph relative to its asymptote, consider the sign of $f(x) - y$. It has the same sign as $\dfrac{a_2}{x}$; therefore,

- near $+\infty$ (respectively $-\infty$), the sign of $f(x) - y$ is the sign of a_2 (respectively $-a_2$).

Example 5.7.1.

1. *Compute the generalized expansion at $+\infty$ of*

$$f(x) = \frac{x^2 - x}{1 + x},$$

and deduce the equation of the asymptote of the graph G_f at $+\infty$, as well as the position of the graph relative to that line.

Let $y = \dfrac{1}{x}$. Then we have

$$f\Big(\frac{1}{y}\Big) = \frac{\Big(\frac{1}{y}\Big)^2 - \frac{1}{y}}{1 + \frac{1}{y}} = \frac{\frac{1-y}{y^2}}{\frac{1+y}{y}} = \frac{1 - y}{y(1 + y)}.$$

We expand $\dfrac{1 - y}{1 + y}$ as a series around $y = 0$:

$$\frac{1 - y}{1 + y} = (1 - y)(1 - y + y^2 - \dots) = 1 - 2y + 2y^2 + o(y^2), \quad y \to 0^+.$$

Hence,

$$f\Big(\frac{1}{y}\Big) = \frac{1 - 2y + 2y^2 + o(y^2)}{y} = \frac{1}{y} - 2 + 2y + o(y), \quad y \to 0^+,$$

which corresponds to

$$f(x) = x - 2 + \frac{2}{x} + o\Big(\frac{1}{x}\Big), \quad x \to +\infty.$$

Then

$$f(x) = x - 2 + \frac{2}{x} + o\Big(\frac{1}{x}\Big).$$

Since

$$\lim_{x \to +\infty} \big(f(x) - (x - 2)\big) = \lim_{x \to +\infty} \Big(\frac{2}{x} + o\Big(\frac{1}{x}\Big)\Big) = 0,$$

the asymptote is the line $y = x - 2$. To find the position of the graph, consider

$$f(x) - (x-2) = \frac{2}{x} + o\Big(\frac{1}{x}\Big).$$

This difference is positive near $+\infty$; hence the graph lies above its asymptote as $x \to +\infty$.

2. *Compute the generalized expansion at $-\infty$ of*

$$f(x) = \sqrt{1 + x + x^2},$$

and deduce the asymptote of the graph G_f at $-\infty$, as well as the position of the graph relative to that line.

Let $y = \dfrac{1}{x}$. Then

$$f\Big(\frac{1}{y}\Big) = \sqrt{1 + \frac{1}{y} + \frac{1}{y^2}} = \frac{1}{y}\big(1 + y + y^2\big)^{1/2} = \frac{1}{y} + \frac{1}{2} + \frac{3}{8}y + o(y).$$

Hence, in terms of x,

$$f(x) = x + \frac{1}{2} + \frac{3}{8x} + o\Big(\frac{1}{x}\Big),$$

so $y = x + \frac{1}{2}$ is an asymptote at $-\infty$. Since

$$f(x) - \Big(x + \frac{1}{2}\Big) = \frac{3}{8x} + o\Big(\frac{1}{x}\Big),$$

this difference is negative near $-\infty$; therefore the graph lies below its asymptote as $x \to -\infty$.

Chapter 6

Sequences of Real Numbers

6.1 Definitions and General Properties

Definition 6.1.1. *A* sequence of real numbers *is any function from* $\mathbb{N}$ *to* $\mathbb{R}$. *Typically, a sequence is denoted by* $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}, \ldots$. *For a sequence* $(u_n)_{n\in\mathbb{N}}$, *the number* u_n *is called the* general term of rank n.

Example 6.1.1. *Consider the sequences defined by*

$$(u_n) = \left(\frac{n+1}{n+2}\right), \quad (v_n) = \big((-1)^n\big), \quad (x_n) = (a).$$

Compute the terms u_0, u_1, *and* u_2.

Definition 6.1.2. *A sequence* $(u_n)_{n\geq 0}$ *is said to be:*

1. increasing *if* $\forall n \in \mathbb{N} : u_{n+1} \geq u_n$,
2. decreasing *if* $\forall n \in \mathbb{N} : u_{n+1} \leq u_n$,
3. monotone *if it is either increasing or decreasing,*
4. bounded above *if* $\exists M \in \mathbb{R}, \forall n \in \mathbb{N} : u_n \leq M$,
5. bounded below *if* $\exists m \in \mathbb{R}, \forall n \in \mathbb{N} : u_n \geq m$,

6. bounded *if it is both bounded above and below,*

7. constant *if* $\forall n \in \mathbb{N} : u_{n+1} = u_n$.

Monotonicity can be determined by studying the sign of $u_{n+1} - u_n$. If u_n is positive, one can also compare the ratio $\dfrac{u_{n+1}}{u_n}$ with 1, or, if $u_n = f(n)$, use the functional approach.

Exercise 6.1.1.

1. *Consider the sequence* $(u_n) = ((-1)^n)$. *Is it bounded? Monotone?*
   We have $u_{2n} = 1$ *and* $u_{2n+1} = -1$, *so* (u_n) *is not monotone because it is alternating. Since* $-1 \leq u_n \leq 1$, *the sequence* (u_n) *is bounded.*

2. *Consider the sequence* $(v_n)_{n\geq 2}$ *defined by*

$$v_n = \frac{n+2}{n-1}.$$

   (a) *Show that* $(v_n)_{n\geq 2}$ *is decreasing.*

   (b) *Deduce that* $(v_n)_{n\geq 2}$ *is bounded.*

6.2 Convergence of Sequences

Definition 6.2.1. *Let* $(u_n)_{n\geq 0}$ *be a sequence of real numbers. We say that* (u_n) converges to l *if*

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0 :\ |u_n - l| < \epsilon.$$

We write $\lim\limits_{n \longrightarrow +\infty} u_n = l$.
A sequence that does not converge is called divergent, *i.e., either* $\lim\limits_{n \longrightarrow +\infty} u_n = \infty$ *or the limit does not exist.*

Theorem 6.2.1. *If a sequence $(u_n)_{n\geq 0}$ converges, then it is bounded.*

Remark 6.2.1. *The converse of the theorem is false. There exist bounded sequences that do not converge. For example, $u_n = (-1)^n$ is bounded but not convergent, because its limit does not exist ($u_{2n} = 1$ and $u_{2n+1} = -1$).*

Theorem 6.2.2. *Let (u_n) and (v_n) be sequences such that (u_n) is bounded and (v_n) converges to 0. Then the sequence $(u_n v_n)$ converges to 0.*

Example 6.2.1.

$$\lim_{n\longrightarrow+\infty} \frac{(-1)^n}{n} = 0$$

because the sequence $(-1)^n$ is bounded and $\lim_{n\longrightarrow+\infty} \frac{1}{n} = 0$.

Theorem 6.2.3 (Squeeze Theorem)**.**
Let (u_n) and (v_n) be two sequences converging to the same limit l, and let (w_n) be a sequence satisfying

$$u_n \leq w_n \leq v_n$$

for all n sufficiently large. Then (w_n) converges to the same limit l.

Example 6.2.2. *We have*

$$\lim_{n\longrightarrow+\infty} \frac{\sin(n)}{n} = 0$$

because

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$$

and $\lim_{n\longrightarrow+\infty} \frac{1}{n} = 0$.

Theorem 6.2.4 (Comparison Theorem).

Let (u_n) and (v_n) be sequences such that $u_n \leq v_n$ for all n sufficiently large. Then:

1. *If $\lim\limits_{n \longrightarrow +\infty} u_n = +\infty$, then $\lim\limits_{n \longrightarrow +\infty} v_n = +\infty$.*
2. *If $\lim\limits_{n \longrightarrow +\infty} v_n = -\infty$, then $\lim\limits_{n \longrightarrow +\infty} u_n = -\infty$.*

Example 6.2.3. *Consider the sequence*

$$2\cos(n) + 3(-1)^n - 3n.$$

We have

$$2\cos(n) + 3(-1)^n - 3n \leq 5 - 3n$$

and $\lim\limits_{n \longrightarrow +\infty} (5 - 3n) = -\infty$, so $\lim\limits_{n \longrightarrow +\infty} [2\cos(n) + 3(-1)^n - 3n] = -\infty$.

Theorem 6.2.5 (Convergence of Monotone Sequences).

Let $E = \{u_n \mid n \in \mathbb{N}\}$.

Any sequence $(u_n)_{n \geq 0}$ that is increasing and bounded above converges to its supremum:

$$\lim_{n \longrightarrow +\infty} u_n = \sup E.$$

Any sequence $(u_n)_{n \geq 0}$ that is decreasing and bounded below converges to its infimum:

$$\lim_{n \longrightarrow +\infty} u_n = \inf E.$$

Example 6.2.4. *Study the convergence of the sequence*

$$u_n = \frac{n-1}{3n+5}, \quad n \in \mathbb{N}.$$

The sequence $(u_n)_{n \geq 0}$ is increasing because

$$u_{n+1} - u_n = \frac{8}{(3n+5)(3n+8)} > 0$$

and it is bounded above because for $n \neq 0$,

$$3n+5 > 3n \implies \frac{1}{3n+5} < \frac{1}{3n} \implies \frac{n-1}{3n+5} < \frac{n-1}{3n} < \frac{1}{3}.$$

For $n = 0$, $u_0 = -\frac{1}{5} < \frac{1}{3}$. Hence, (u_n) is convergent toward its supremum. Therefore,

$$\lim_{n \longrightarrow +\infty} u_n = \lim_{n \longrightarrow +\infty} \frac{n-1}{3n+5} = \frac{1}{3}.$$

6.3 Recursive Sequences

Definition 6.3.1. *Let $f : I \longrightarrow \mathbb{R}$ and $u_0 \in I$ (we assume $f(I) \subset I$). A* recursive sequence *is a sequence defined by*

$$\begin{cases} u_0 & \textit{given}, \\ u_{n+1} = f(u_n), & \forall n \in \mathbb{N}. \end{cases}$$

6.3.1 Monotonicity

If f is increasing on I, then $\begin{cases} (u_n)_{n\geq 0} \text{ is increasing if } u_1 \geq u_0. \\ (u_n)_{n\geq 0} \text{ is decreasing if } u_1 \leq u_0. \end{cases}$

If f is decreasing on I, then the sequence $(u_n)_{n\geq 0}$ is not monotone. Indeed, assuming $u_1 \geq u_0$, we obtain

$$\begin{aligned} u_1 \geq u_0 &\implies u_2 = f(u_1) \leq f(u_0) = u_1 \\ &\implies u_3 \geq u_2 \\ &\implies u_4 \leq u_3 \text{ (stop)}. \end{aligned}$$

This alternating order $u_4 \leq u_3 \geq u_2 \leq u_1 \geq u_0$ shows that $(u_n)_{n\geq 0}$ is not monotone.

Examples 6.3.1.

1. *The sequence defined by*

$$\begin{cases} u_0 \quad \textit{given}, \\ u_{n+1} = u_n + r, \quad r \in \mathbb{R} \end{cases}$$

is called an arithmetic sequence*. The number r is called the* common difference*. The general term can be written as*

$$u_n = u_0 + nr,$$

or more generally

$$u_n = u_p + (n-p)r, \quad n, p \in \mathbb{N}.$$

The sum of consecutive terms of an arithmetic sequence is

$$S = u_p + u_{p+1} + \cdots + u_m = \frac{m-p+1}{2}(u_p + u_m).$$

Example: *Compute the sum*

$$S = 11 + 13 + 15 + \cdots + 47 + 49$$

of 20 consecutive terms of an arithmetic sequence with common difference 2:

$$S = \frac{20}{2}(11+49) = 600, \quad (20 = \frac{49-11}{2} + 1).$$

2. The sequence defined by

$$\begin{cases} v_0 \quad \textit{given}, \\ v_{n+1} = qv_n, \quad q \in \mathbb{R} \end{cases}$$

is called a geometric sequence. *The number q is called the* common ratio. *The general term can be written as*

$$v_n = v_0 q^n,$$

or more generally

$$v_n = v_p \cdot q^{n-p}, \quad n, p \in \mathbb{N}.$$

The sum of consecutive terms of a geometric sequence is

$$S = v_p + v_{p+1} + \cdots + v_m = v_p \frac{1 - q^{m-p+1}}{1-q}, \quad q \neq 1.$$

Example: *Compute the sum*

$$S = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{16384}$$

of 13 consecutive terms of a geometric sequence with ratio $\frac{1}{2}$ and first term $\frac{1}{4}$:

$$S = \frac{1}{4} \cdot \frac{1 - (\frac{1}{2})^{13}}{1 - \frac{1}{2}} = \frac{8191}{16384}.$$

(Note: $(\frac{1}{2})^{n-1} = \frac{v_n}{v_1} = \frac{1}{4096} \implies n - 1 = \frac{\ln 4096}{\ln 2} = 12.$)

6.3.2 Convergence

Suppose f is monotone and continuous on D. If the sequence (u_n) converges to $l \in D$, then this limit satisfies

$$l = f(l).$$

If this equation has no solution, the sequence has no limit. Otherwise, if there exists one or more numbers l_0 such that $l_0 = f(l_0)$, the problem reduces to determining whether (u_n) converges to one of these l_0.

Exercise 6.3.1. *Consider the sequence $(u_n)_n$ defined by*

$$\begin{cases} u_0 = 2, \\ u_{n+1} = 5 - \dfrac{4}{u_n}, \quad n \in \mathbb{N}. \end{cases}$$

1. *Show that* $\forall n \in \mathbb{N},\ 1 < u_n < 4.$
2. *Show that $(u_n)_n$ converges and find its limit.*
3. *Give* $\sup E$ *and* $\inf E$*, where $E = \{u_n : n \in \mathbb{N}\}$.*

Solution:

1. *By induction. For $n = 0$, we have $1 < u_0 = 2 < 4$, so the property holds. Assume $1 < u_n < 4$ and show $1 < u_{n+1} < 4$ for $n \geq 0$:*

$$\begin{aligned} 1 < u_n < 4 &\Longrightarrow \frac{1}{4} < \frac{1}{u_n} < 1 \\ &\Longrightarrow -4 < -\frac{4}{u_n} < -1 \\ &\Longrightarrow 1 < u_{n+1} = 5 - \frac{4}{u_n} < 4. \end{aligned}$$

Alternatively, consider the function $f(x) = 5 - \dfrac{4}{x}$ *for* $x \in]1,4[$. *Since* f *is increasing and* $f(]1,4[) =]1,4[$, *and* $u_0 = 2 \in]1,4[$, *it follows that* $u_n \in]1,4[$ *for all* $n \geq 0$.

2. *Since* f *is increasing and* $u_1 = 3 > u_0 = 2$, *the sequence* $(u_n)_n$ *is increasing.*

3. *Since* $(u_n)_n$ *is increasing and bounded above by 4, it converges. Let* l *be the limit of* $(u_n)_n$. *Then* l *satisfies*

$$l = 5 - \frac{4}{l} \iff l = 1 \quad or \quad l = 4.$$

Since $(u_n)_n$ *is increasing, we must have* $l = 4$.

4. *Finally, since* $(u_n)_n$ *is increasing and converges to* l,

$$\inf E = u_0 = 2, \quad \sup E = l = 4.$$

6.4 Subsequences

Definition 6.4.1. *Let* (u_n) *be a sequence and* g *a strictly increasing function from* $\mathbb{N}$ *to* $\mathbb{N}$. *Then the sequence* $(u_{g(n)})$ *is called a* ***subsequence*** *of* (u_n).

Examples 6.4.1.

1) The sequences $(u_{2n}) = (u_0, u_2, ..., u_{2n}, ...)$ *and* $(u_{2n+1}) = (u_1, u_3, ..., u_{2n+1}, ...)$ *are two subsequences of* $(u_n) = (u_0, u_1, ..., u_n, ...)$.
For instance, the sequences $(u_{2n}) = (1, 1, ..., 1, ...)$ *and* $(u_{2n+1}) = (-1, -1, ..., -1, ...)$ *are two subsequences of* $(u_n) = ((-1)^n)$ *with* $g_1(n) = 2n$ *and* $g_2(n) = 2n+1$.
2) The sequences $(u_{n^3}) = \left(\dfrac{1}{n^3}\right)$ *and* $(u_{2^n}) = \left(\dfrac{1}{2^n}\right)$ *are subsequences of* $(u_n) = \left(\dfrac{1}{n}\right)$ *with* $g_1(n) = n^3$ *and* $g_2(n) = 2^n$.

Theorem 6.4.1. *If the sequence (u_n) converges to l, then all subsequences of (u_n) also converge to l.*

Remark 6.4.1. *The converse is false: it is possible to find a convergent subsequence of a divergent sequence.*
Example: $(u_n) = ((-1)^n)$ is divergent, but its two subsequences $(u_{2n}) = (1)$ and $(u_{2n+1}) = (-1)$ converge.

Theorem 6.4.2. *A sequence (u_n) converges to l if and only if the two subsequences (u_{2n}) and (u_{2n+1}) converge to the same limit l.*

In other words, if a subsequence of (u_n) diverges, then (u_n) diverges; or if two subsequences have different limits, then (u_n) diverges.
Example: If $\lim\limits_{n \longrightarrow +\infty} u_{2n} = 1 \neq \lim\limits_{n \longrightarrow +\infty} u_{2n+1} = -1$, then $(u_n) = ((-1)^n)$ diverges and $\lim\limits_{n \longrightarrow +\infty} (-1)^n$ does not exist (because it cannot tend to infinity).

Exercise 6.4.1. *Show that $\lim\limits_{n \longrightarrow +\infty} \sin\left(\dfrac{n\pi}{2}\right)$ does not exist.*

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