République Algérienne Démocratique et Populaire
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique
École Supérieure d'Économie d'Oran

Polycopié pédagogique

Analysis-I-

Course summary and solved exercises

destinés aux étudiants de (première année / cycle préparatoire)

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Année universitaire 2023 / 2024

About Analysis

The present course deals with the most basic concepts in analysis. The goal of the course is to acquaint the reader with rigorous calculus and application of theorems and propositions in analysis.

We start with a discussion of the real number, most importantly its completeness property, which is the basis for all that comes after. We then discuss the simplest form of a limit, the limit of a sequence. Afterwards, we study functions of one variable, continuity, and the derivative, ending with the chain rule and Taylor's theorem are discussed. Next, we discuss sequences of real valued. We define the definite and indefinite integrals. Finally, we give the fundamental ruls of calculus to obtain the primitive of real-valued functions of one variables.

Preface

This handout is intended for students of the first year of Higher Graduate School of Economics. This handout deals with an important part of Mathematical Analysis is compliant the official program of Mathematical Analysis for the Higher Schools of Economics. It should be possible to use the handout for both a basic course for students who do not necessarily wish to go to graduate school (such as Higher Graduate School of Economics), but also as a more advanced two-semester course that also covers topics such as science and technology, a high school teacher.

This programme will be covered by the nine chapters, each chapter in sub-chapters, each sub-chapter in paragraphs, numbered in the order of this division, so that the reader can quickly find a part quoted in the presentation.

The exercises are classified by chapters corresponding to the major divisions of the program. Within each chapter, each of the subdivisions is preceded by a reminder of the essence of the corresponding part.

This course has been written in such a way that the essentials are said for each concept introduced. Are there to help the student position the acquisition of his knowledge (concepts and results) in a "broad line", they give the theorems and essential results without demonstrations

This course contains many examples illustrating the definitions and helping the understanding of the conditions of application of theorems and propositions.

The drafting of the solutions was done with the greatest care, the main concern being to familiarize the student with mathematical reasoning and him. learn especially to write.

We hope that the practice of this handout will help students to asimile more quickly the concepts and methods introduced to the course. Our greatest thanks go to all those who have contributed to the training of students at the Higher Graduate School of Economics.

Finally, it is with gratitude that we welcome criticism and Suggestions that users of this handout will send us.

Notation

In this handout, we will use the following notations.

Symbols logiques et mathématiques:
$\in$ in,$x \in A$ *i.e.*, x is an element of A.
$\subset$ inclusion,
$\leq$, $\geq$ large inequalities, $x \leq y$, x less than or equal to y. $x \geq y$ read x is greater than or equal to y.
$<$, $>$ strict inequality, $x < y$ read x strictly less than y. $x > y$ read x strictly greater than y.
$\Rightarrow$ implication, $p \Rightarrow q$ read p implies q(if p then q).
$\Longleftrightarrow$ equivalence, $p \Longleftrightarrow q$ read p is quivalent to q. (p if and only if q).
$E(x)$ or $[x]$: integer part of x, $x \in \mathbb{R}$.
C_n^k: Coefficient of binom, cardinality of the set of subsets of k elements of a set of n elements, $k \leq n$.
$R[X]$ The set of plynomials with real coefficients of inconnu X.
$\oslash$ empty set.
$|\cdot|$ absolute value.
x^y exponentiation of $x > 0$ and $y \in \mathbb{R}$.
f', f'', $f^{(3)}, \ldots, f^{(n)}$ second, third,..., n^{th} derivative of f.
$\sup E$ supremum of E.
$\inf E$ infimum of E.
$\max E$ maximum of E.
$\min E$ minimum of E.
$]a, b[$ open bounded interval
$[a, b]$ closed bounded interval
$]a, b]$, $[a, b[$ half-open bounded interval or half-closed bounded interval
$\mathbb{R}^*$the extended real numbers
$\overline{\mathbb{R}} = [-\infty, +\infty]$
∞ infinity.

Alphabet grec:
α alpha.
β bêta.
γ gamma.
δ, $\triangle$ delta.
ϵ epsilon.
λ lambda.
θ têta.
$\sum$ sigma.

Chapter 1

Real numbers

1.1 Basic sets

The following are sets including the standard notations. We donote the set of natural numbers by

$$\mathbb{N} = \{0, 1, 2, \dots, n, \dots\},$$

the set of even natural numbers and the set of odd natural numbers respevtively

$$\{2m, \quad m \in \mathbb{N}\}, \ and \ \{2m+1, \quad m \in \mathbb{N}\},$$

and the set of integers by

$$\mathbb{Z} = \{\dots, -n, \dots, -2, -1, 0, 1, 2, \dots, n, \dots\},$$

and the set of rational numbers by

$$\mathbb{Q} = \left\{ r = \frac{p}{q},\ p,\ q \in \mathbb{Z},\ q \neq 0 \right\}$$
$$= \left\{ \frac{p}{q},\ p \in \mathbb{Z},\ q \in \mathbb{N}^* \right\}.$$

We show that all rationnel number $\frac{p}{q}$ can be written at the decimal limit expansion form as

$$\frac{p}{q} = \alpha_0 \alpha_1 \dots \alpha_n,$$

or in periodic infinitely form

$$\frac{p}{q} = \alpha_0 \alpha_1 \dots \alpha_n \beta_1 \beta_2 \dots \beta_m \beta_1 \beta_2 \dots \beta_m \dots,$$

where $\alpha_0 \in \mathbb{N},\ \alpha_k,\ \beta_j \in \{0, 1, 2, \dots, 9\}$ et $\beta_1 \beta_2 \dots \beta_m$ is the period.

Remark 1. Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Definition 1. We called an irrational number every dicimal limit expansion periodic.

Definition 2. $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}$, where $\mathbb{Q}^C$ is the complement of $\mathbb{Q}$.

1.2 Axiomatic definition of $\mathbb{R}$

The set $\mathbb{R}$ endowed (equiped) by two binary operations : Addition $+$ and Multiplication $\times$ or $\cdot$ and order relation $\leq$ satisfying the following axioms:

(A_1) Commutativity : The commutative law holds for addition $x+y=y+x,\ \forall x,\ y\in\mathbb{R}$.

(A_2) Associativity : The associative law holds for addition $x+(y+z)=(x+y)+z,\ \forall x,\ y,\ z\in\mathbb{R}$.

(A_3) Additive identity element e : $\exists e=0\in\mathbb{R}\,/\,x+e=e+x=x,\ \forall x\in\mathbb{R}$.

(A_4)Multiplicative identity element e : $\exists e=1\in\mathbb{R}\,/\,x\cdot e=e\cdot x=x,\ \forall x\in\mathbb{R}$.

(A_5) Additive inverse element $x'=-x$ for x: $\forall x\in\mathbb{R},\ \exists x'=-x\in\mathbb{R}\,/\,x+x'=0$.

(A_6) $x\cdot y=y\cdot x,\ \forall x,\ y\in\mathbb{R}$.

(A_7) Associativity: associative law holds for multiplication $x\cdot(y\cdot z)=(x\cdot y)\cdot z, \forall x,\ y,\ z\in\mathbb{R}$.

(A_8) Multiplicative inverse element $x'=\frac{1}{x}=x^{-1}$ de x: $\forall x\in\mathbb{R},\ x\neq 0,\ \exists x'=\frac{1}{x}\in\mathbb{R}\,/\,x\cdot x'=1$.

(A_9) Distributivity: The distributive law holds: $x\cdot(y+z)=x\cdot y+x\cdot z,\ \forall x,\ y,\ z\in\mathbb{R}$.

(A_{10}) $x\leq x,\quad \forall x\in\mathbb{R}$.

$(A_{11})$$\forall x,\ y\in\mathbb{R}:\ (x\leq y\ \wedge\ y\leq x)\Longrightarrow x=y$.

(A_{12}) Transitivity : ,$\forall x,\ y,\ z\in\mathbb{R}:\ (x\leq y\ \wedge\ y\leq z)\Longrightarrow x\leq z$.

(A_{13}) Ordred $\forall x,\ y\in\mathbb{R},$ *on a soit* $x\leq y$ *soit* $y\leq x$.

(A_{14}) $\forall x,\ y\in\mathbb{R}:\ x\leq y\Longrightarrow x+z\leq y+z,\ \forall z\in\mathbb{R}$.

(A_{15}) $\forall x,\ y\in\mathbb{R}:\ x\leq y,\ z\geq 0\Longrightarrow x\cdot z\leq y\cdot z$.

1.3 Absolute value

Definition 3. We define the absolute value of $x\in\mathbb{R}$ by

$$|x|=\begin{cases} x, & if\ x\geq 0,\\ -x, & if\ x<0.\end{cases}$$

1.4 Intervals of $\mathbb{R}$

Definition 4. The set I of $\mathbb{R}$ is called an interval if and only if satisfying the following properties

$$x,\ y\in I,\ \ x<y\Longrightarrow \forall z\in\mathbb{R}\ :\ (x<z<y\Rightarrow z\in I)\,.$$

The different intervals in $\mathbb{R}$ sont: Let $a,\ b\in\mathbb{R}$ such that $a<b$.

1) Closed interval : $[a,b]=\{x\in\mathbb{R}/\ a\leq x\leq b\}$ (Bounded interval of extremities a and b, both end points are included).

2) Open interval : $]a,b[=\{x\in\mathbb{R}/\ a<x<b\}$ (Bounded interval of extremities a and b, neither end point is included.).

3) Half-open interval (Half-closed) : $[a,b[=\{x\in\mathbb{R}/\ a\leq x<b\}$ (Bounded interval of extremities a and b).

4) Half-open interval (Half-closed) : $]a,b]=\{x\in\mathbb{R}/\ a<x\leq b\}$ (Bounded interval of extremities a and b).

5) $[a, +\infty[= \{x \in \mathbb{R} / \ x \geq a\}$ (unbounded intervals of extremity a).
6) $]a, +\infty[= \{x \in \mathbb{R} / \ x > a\}$ (unbounded intervals of extremity a).
7) $]-\infty, b] = \{x \in \mathbb{R} / \ x \leq b\}$ (unbounded intervals of extremity b).
8) $]-\infty, b[= \{x \in \mathbb{R} / \ x < b\}$ (unbounded intervals of extremity b).
10) $\mathbb{R} =]-\infty, +\infty[$ (Open and closed interval of $\mathbb{R}$).

1.5 Supremum and infimum of a Set of $\mathbb{R}$

Definition 5. We say that $X \subset \mathbb{R}$ is

1) bounded above if there exists $M \in \mathbb{R}$ called upper bound of X such that $\forall x \in X$, $x \leq M$,
2) bounded below if there exist $m \in \mathbb{R}$ called lower bound of X such that $\forall x \in X$, $x \geq m$,
3) bounded if there exist M, $m \in \mathbb{R}$ such that $\forall x \in X$, $m \leq x \leq M$.

Remark 2. (i) Any larger number of X called upper bound of X and any smaller number of X called lower bound of X.

(ii) If M is an upper bound of X, but no number less than M, then M is a supremum of X, and we write $\sup X = M$.

(iii) If m is an lower bound of X, but no number greater than m, then m is a infimum of X, and we write $\inf X = m$.

(v)If there exists $x_0 \in X$ such that $M = x_0$ or $m = x_0$, then $M = \max X$ or $m = \min X$.

Theorem 1. *1) A bounded nonempty set has a unique supremum and a unique infimum.*

2) If a nonempty set of real numbers is bounded above (respectively bounded below), then it has a supremum (respectively infimum).

Remark 3. The property (2) of Theorem 1, called property of the upper bound , it is not true in the set of rational numbers $\mathbb{Q}$.

Theorem 2. *1) If a nonempty set X of real numbers is bounded above and $M \in \mathbb{R}$, then*

$$M = \sup X \Longleftrightarrow \begin{cases} i) \ M \ is \ an \ upper \ bound \ of \ X, \\ ii) \ \forall \epsilon > 0, \ \exists x \in X \ : \ M - \epsilon < x \leq M. \end{cases}$$

2) If a nonempty set X of real numbers is bounded below and $m \in \mathbb{R}$, then

$$m = \inf X \Longleftrightarrow \begin{cases} i) \ m \ is \ an \ lower \ bound \ of \ X, \\ ii) \ \forall \epsilon > 0, \ \exists x \in X \ : \ m \leq x < m + \epsilon. \end{cases}$$

1.6 Archimedean property and their consequences.

The following property called Archimedean property

$$\forall x \in \mathbb{R}, \ \exists n \in \mathbb{Z} \ : \ n > x.$$

This property is also often expressed as

$$\forall h \in \mathbb{R},\ \forall x \in \mathbb{R},\ \exists n \in \mathbb{Z} : nh > x.$$

We deduce

$$\forall x \in \mathbb{R},\ \exists n \in \mathbb{Z} : n \leq x < n+1.$$

Definition 6. The integer number $n \in \mathbb{Z}$ satisfying the above relation is called the integer patrt of x denoted by $[x]$ or $E(x)$.

The integer number n is less than x.

For example: $[0,28] = E(0,28) = 0$, $\left[\sqrt{2}\right] = 1$, $[-1,20] = -2$.

We deduce from the Archemedean property, the following theorem :

Theorem 3. *(Dense property): (i) For all real numbers a, b such that $a < b$, there is rationnal number r such that $a < r < b$ that is, the set of rational numbers is dense in the reals.*

Lemma 1. (ii) *For all real numbers a, b such that $a < b$, there is rationnal number s such thata $< s < b$ that is the set of irrational numbers is dense in the reals.*

1.7 Demonstration by reccurence (Principle of Mathematical Induction)

Let $p(n)$ be proposition for each positive integer $n \in \mathbb{N}$ ($P(n)$ be a statement depending on a natural number n). Suppose that

(1) (Basis statement) $p(n_0)$ is true.

(2) (Induction step) If $p(n)$ is true for each positive integer $n \geq n_0$, then $p(n+1)$ is true.

Then $p(n)$ is true $\forall n \geq n_0$.

1.8 Topology elements in $\mathbb{R}$

1.8.1 Open and closed sets in $\mathbb{R}$

In this section we study topological concepts in the context of the real line.

Definition 7. (i) The set $O \subset \mathbb{R}$ is called open if $O = \emptyset$ or if for all $x \in O$, there is an open interval $I \subset \mathbb{R}$ contain in O and containing the point x that is we say that a set is open if it does not contain any of its boundary points.

(ii) The set $F \subset \mathbb{R}$ is called closed if it's complement in $\mathbb{R}$ is open that is Aa set F is said to be closed if it contains all its boundary points.

Remark 4. Note that $\mathbb{R}$ and $\emptyset$ are both open and closed.

Example 1. 1) The intervals $]a,b[\,,\ ]-\infty,a[\,,\ ]a,+\infty[$ *and* $]-\infty,+\infty[$ are opens.

2) The intervals $]-\infty,a]\,,\ [a,+\infty[$ *and* $]-\infty,+\infty[$ are closed.

1.8.2 Notion of neighborhood of a point.

Definition 8. (i) Let $x \in \mathbb{R}$. We say that a neighborhood of a point x is any subset V de $\mathbb{R}$ containing an open O contain it self a point x.

(ii) A neighborhood of a point $a \in \mathbb{R}$ is an interval of the form $]a-\epsilon, a+\epsilon[$, where $\epsilon > 0$ is any positive real number. Thus, the neighborhood consists of all points distance less than ϵ from a:

$$]a-\epsilon, a+\epsilon[= \{x \in \mathbb{R} : |x-a| < \epsilon\}.$$

Example 2. 1) The open interval $I =]a, b[$ is a neighborhood for any $x \in I$ and any set of the form $]x-\delta, x+\delta[$, $\delta > 0$, is called neighborhood of x.

2) The closed interval $E = [a, b]$ is a neighborhood for any point $x \in]a, b[$, but it does't a neighborhood of a points a and b. Indeed, $\forall x \in]a, b[$, we have $]a, b[\subset E$ and $x \in]a, b[$, but, according to the definition of the interval there is no open interval contains a or b and include in $[a, b]$.

1.9 $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 9. We say that the set $A \subset \mathbb{R}$ is dense in $\mathbb{R}$ if $\overline{A} = \mathbb{R}$. That is

$$\overline{A} = \mathbb{R} \Longleftrightarrow \forall a,\, b \in \mathbb{R} : (a < b \Longrightarrow]a, b[\cap A = \emptyset).$$

.

Lemma 2. *The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$, that is $\overline{\mathbb{Q}} = \mathbb{R}$.*

Exercise 1. Resolve in $\mathbb{R}$ the following equations:

1)$|3x-4| = \frac{1}{2}$, 2) $|x+1| - |x-1| = 2$, 3) $\sqrt{(x-2)^2} = -x+2$

solution: We write the following expressions in equivalent forms not involving absolute values:

1) From the properties of absolute value, we have

$$|3x-4| = \frac{1}{2} \Longleftrightarrow \begin{cases} 3x-4 = \frac{1}{2},\ if\ 3x-4 \geq 0, \\ 3x-4 = -\frac{1}{2},\ if\ 3x-4 < 0, \end{cases}$$

$$\Longrightarrow \begin{cases} x = \frac{3}{2},\ if\ x \geq \frac{4}{3}, \\ x = \frac{7}{6},\ if\ x < \frac{4}{3}, \end{cases}$$

Thus, the set of solutions of the equation is $S = \left\{\frac{7}{6}, \frac{3}{2}\right\}$.

2) Table of the sign

x	-1		1
$\lvert x+1\rvert$	$-x-1$	$x+1$	$x+1$
$\lvert x-1\rvert$	$-x+1$	$-x+1$	$x-1$
$\lvert x+1\rvert - \lvert x-1\rvert = 2$	-2=2 fausse	$2x = 2 \Rightarrow x = 1$	2=2 vraie

Thus, the set of the solutions of the equation is $S = [1, +\infty[$.

3) From the properties of absolute value, we have $\sqrt{(x-2)^2} = |x-2|$. So

$$\begin{aligned}
\sqrt{(x-2)^2} = -x+2 &\Longleftrightarrow |x-2| = -x+2 \\
&\Longleftrightarrow \begin{cases} x-2 = -x+2, & if\ x-2 \geq 0, \\ -x+2 = -x+2, & if\ x-2 < 0, \end{cases} \\
&\Longleftrightarrow \begin{cases} 2x-4 = 0, & if\ x \geq 2, \\ 2 = 2, & if\ x < 2, \end{cases} \\
&\Longleftrightarrow \begin{cases} x = 2, & if\ x \geq 2, \\ 2 = 2, & if\ x < 2, \end{cases}
\end{aligned}$$

Thus, the set of the solutions of the equation is $S =]-\infty, 2]$.

Exercise 2. Resolve in $\mathbb{R}$ the following inequalities :
1) $|x+1| < 1$, 2) $|x+2| \geq 5$, 3) $|2x-1| < |x-1|$.

Solution: By the help of the properties of absolute value the inequality can be expressed in another form not involving absolute value :

To solve the inequality, we will must express my answer in interval notation.

1) From the properties of absolute value, we have

$$\begin{aligned}
|x+1| < 1 &\Longleftrightarrow -1 < x+1 < 1 \\
&\Longleftrightarrow -2 < x < 0 \\
&\Longleftrightarrow x \in]-2, 0[\,.
\end{aligned}$$

Thus, the set of the solutions of the inequality is$S =]-2, 0[$.

2) From the properties of absolute value, we have

$$|x+2| \geq 5 \Longleftrightarrow \begin{cases} x+2 \geq 5,\ if\ x+2 \geq 0, \\ \quad or \\ x+2 \leq -5,\ if\ x+2 < 0, \end{cases}$$

$$\Longleftrightarrow \begin{cases} x \geq 3 \quad if\ x \geq 2, \\ \quad or \\ x \leq -7,\ if\ x < 2. \end{cases}$$

Thus, the set of the solutions of the inequality is $S =]-\infty, -7] \cup [3, +\infty[$.

3) We have $|2x-1| < |x-1| \Longleftrightarrow |2x-1| - |x-1| < 0$.

Table of the sign

x	$\frac{1}{2}$		1
$\|2x-1\|$	$-2x+1$	$2x-1$	$2x-1$
$\|x-1\|$	$-x+1$	$-x+1$	$x-1$
$\|2x-1\| - \|x-1\| < 0$	$x > 0$	$x < \frac{2}{3}$	$x < 0$

Thus, the set of the solutions of the inequality is $S = \left]0, \frac{2}{3}\right[$.

Exercise 3. Find the supremum and infimum of each set. State whether they are in the set :

For each following sets, determine (find) the upper bound, lower bound, the supremum and infimum, maximum, minimum if they exist :

$E_1 = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$, $E_2 = \left\{2 - \frac{1}{n},\ n \in \mathbb{N}^*\right\}$, $E_3 = \left\{\frac{-1+2n}{3+n},\ n \in \mathbb{N}^*\right\}$, $E_4 = \left\{\frac{1}{x},\ 1 \leq x \leq 2\right\}$.

Solution: 1) $E_1 = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$. It is clear that, $\sup E_1 = \max E_1 = 1$ and $\inf E_1 = 0$.

2) It is clear that

$$1 \leq 2 - \frac{1}{n} < 2,\ \forall n \in \mathbb{N}^*.$$

That is, the set E_2 is bounded and 2 is upper bound of E_2. We show that $\sup E_2 = 2$. It is sufficient to prove $\forall \epsilon > 0$, $\exists n_\epsilon \in \mathbb{N}^*\ :\ 2 - \frac{1}{n} > 2 - \epsilon$. So, we obtain $n > \frac{1}{\epsilon}$. For example the first natural n satisfied the inequality $n > \frac{1}{\epsilon}$ (n exists, since $\mathbb{R}$ is archimedien) such that $2 - \frac{1}{n} > 2 - \epsilon$. Therefore, we find $\sup E_2 = 2 \notin E_2$, that is $\max E_2$ not exists. The smallest number of E_2 is 1. Thus, $\min E_2 = \inf E_2 = 1$.

3) $E_3 = \left\{\frac{-1+2n}{3+n},\ n \in \mathbb{N}^*\right\}$. We can represent E_3 in the following form :

$$E_3 = \left\{2 - \frac{7}{3+n},\ n \in \mathbb{N}^*\right\}.$$

So, since $n \in \mathbb{N}^*$, we have $2 - \frac{7}{3+n} < 2$, cthat is the set E_2 is bounded above by 2. We prove that $\sup E_3 = 2$.

$$\sup E_2 = 2 \Longleftrightarrow \begin{cases} \forall n \in \mathbb{N}^*, & 2 - \frac{7}{3+n} \leq 2, \\ \forall \epsilon > 0,\ \exists n_\epsilon \in \mathbb{N}^*\ :\ & 2 - \frac{7}{3+n_\epsilon} > 2 - \epsilon. \end{cases}$$

We choose n_ϵ from the inequality $2 - \frac{7}{3+n} > 2 - \epsilon$. Thus, we have $n > \frac{7}{\epsilon} - 3$.
Then, we can take the natural n satisfied the inequality $n > \frac{7}{\epsilon} - 3$. Thus, we have $\sup E_3 = 2$. It is clear that $\forall n \in \mathbb{N}^*\ :\ \frac{-1+2n}{3+n} \geq \frac{1}{4}$ and for $n = 1$, we have $\frac{-1+2n}{3+n} = \frac{1}{4}$. Then, $\min E_3 = \inf E_3 = \frac{1}{4}$.

4) $E_4 = \left\{\frac{1}{x},\ 1 \leq x \leq 2\right\}$. We have

$$\frac{1}{x} \in \left[\frac{1}{2}, 1\right].$$

So, $\sup E_4 = \max E_4 = 1$ and $\min E_4 = \inf E_4 = \frac{1}{2}$.

Chapter 2

Sequences (Sequences of real numbers)

In this section, we introduce sequences and define what it means for a sequence to converge or diverge. We show how to find limits of sequences that converge, often by using the properties of limits for functions discussed earlier. We close this section with the monotone convergence theorem, a tool we can use to prove that certain types of sequences converge.

2.1 Sequences of real numbers

We have already seen sequences used informally. Let us give the formal definition.

Definition 10. A sequence (of real numbers) is a mapping from $\mathbb{N}$ (or subset of $\mathbb{N}$) to (into) $\mathbb{R}$ that is $u : \underset{n \mapsto u_n}{\mathbb{N} \to \mathbb{R}}$. we usually denote the n^{th} element in the sequence by u_n called the general term of the sequence denoted by $(u_n)_{n \in \mathbb{N}}$ or (u_n). The symbol n is called the index variable for the sequence (u_n).

Example 3. 1) Let $a \in \mathbb{R}$. $u : n \longmapsto a$ is a constant sequence such that $u_n = a, \ \forall n \in \mathbb{N}$.
2) $u : n \longmapsto (-1)^n a$ is alternating sequence such that, we have $u_{2k} = a, \ and \ u_{2k+1} = -a, \ \forall k \in \mathbb{N}$.
3) $u : n \longmapsto a + nb, \ \forall n \in \mathbb{N} \ and \ (a, b) \in \mathbb{R}^2$ is arithmetic sequence of the first term a and a raison (common difference) $r = b$.
4) $u : n \longmapsto a \cdot q^n, \ \forall n \in \mathbb{N} \ and \ (a, q) \in \mathbb{R}^2$ is geometric sequence of the first term a and a raison (common ratio) q.

2.1.1 Recurrents sequences

Definition 11. We called (u_n) an recurrente sequence of order 1 the sequence defined by the first term u_0 and a recurrence relation $u_{n+1} = f(u_n), \ \forall n \in \mathbb{N}$.

Example 4. 1) The sequence defined by the recurrence relation

$$\begin{cases} u_{n+1} = u_n + 1, \ \forall n \in \mathbb{N}, \\ u_0 = 3, \end{cases}$$

is arithmetic sequence of the first term $u_0 = 3$ and the raison (common difference) $r = 1$. Note that, in an arithmetic sequence, the difference between every pair of consecutive terms is the same and equal to 4, in this example, each of the recurring can be obtained from the previous term.

2) The sequence defined by the recurrence relation of the first term $v_0 = 3$,

$$\begin{cases} v_{n+1} = 3v_n, \ \forall n \in \mathbb{N}, \\ v_0 = 3, \end{cases}$$

is geometric sequence of the first term $v_0 = 3$ and the raison (common ratio) $q = 2$.

2.1.2 Monotonic sequences (Monotone sequences)

Definition 12. A sequence of real number (u_n) is :

1) Increasing if : $u_n \leq u_{n+1}, \ \forall n \in \mathbb{N}$.
2) Decreasing if $u_{n+1} \leq u_n, \ \forall n \in \mathbb{N}$.
3) Constante ou stationnaire if : $u_n = u_{n+1}, \ \forall n \in \mathbb{N}$.
4) Monotone if it is either increasing or decreasing.

We should examine a few examples and convince yourself that then a monotone or not.

Example 5. 1) The sequence of real numbers (u_n) defined by $u_n = 3(2)^n, \ \forall n \in \mathbb{N}$ is increasing.

2) The sequence ofvreal numbers (v_n) defined by $u_n = \frac{3}{2^n}, \ \forall n \in \mathbb{N}$ is decreasing.

3) The sequence of real numbers (w_n) defined by $w_n = (-1)^n, \ \forall n \in \mathbb{N}$ is not monotone.

2.1.3 Bounded (bound) sequences

Definition 13. A sequence of real numbers (u_n) is called :

1) Bounded above if there is (exists) a real number M such that : $u_n \leq M, \ \forall n \in \mathbb{N}$.
2) Bounded below if there is (exists) a real number m such that : $u_n \geq m, \ \forall n \in \mathbb{N}$.
3) Bounded if it is bounded above and bounded below, that is, if there is a real numbers $M \, and \, m$ such that :

$$m \leq u_n \leq M, \ \forall n \in \mathbb{N}.$$

Or bounded if there is a real number M such that $|u_n| \leq M, \ \forall n \in \mathbb{N}$.

Remark 5. If a sequence is not bounded, it is an unbounded sequence.

Corollary 1. *(i) A monotone decreasing sequence (u_n) and bounded below converges to* $\lim\limits_{n\to+\infty} u_n = \inf\{u_n, \quad n \in \mathbb{N}\}$.

(ii) A sequence monotone decreasing sequence (u_n) and bounded above converges to $\lim\limits_{n\to+\infty} u_n = \sup\{u_n, \quad n \in \mathbb{N}\}$.

Example 6. 1) The sequence of real numbers (u_n) defined by $u_0 = 3 \, et \, u_n = 1 - u_n, \ \forall n \in \mathbb{N}^*$ is bounded. In fact, $\forall n \in \mathbb{N}^*, \ -2 \leq u_n \leq 3$. Otherwise, the sequence (v_n) defined by $v_n = 3(2)^n, \ \forall n \in \mathbb{N}$ is not bounded (unbounded). In fact $\forall n \in \mathbb{N}, \ v_n \geq 3$, the sequence

is bounded below. However, the sequence is not bounded above. Therefore, (v_n) is an unbounded sequence.
2) The sequence (w_n) of general term $w_n = a \cdot q^n$, $\forall n \in \mathbb{N}$ *et* $(a, q) \in \mathbb{R}^2$ is bounded if $|q| \leq 1$ and it is unbounded if $|q| > 1$.
3) Take (T_n) of general term $T_n = \frac{1}{\sqrt{n}}$. The sequence (T_n) is bounded below as $T_n = \frac{1}{\sqrt{n}} > 0$, $\forall n > 0$. Let us show that it is monotone decreasing. We start with

$$\sqrt{n+1} \geq \sqrt{n} \Rightarrow \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \Leftrightarrow T_{n+1} \leq T_n.$$

That is (This means that), the sequence (T_n) is monotone decreasing and bounded below. Hence bounded. Says that that the sequence is convergent and $\lim\limits_{n\to+\infty} T_n = \inf\{u_n, \quad n \in \mathbb{N}\} = 0$. We already know that the infimum is greater than or equal to 0, as 0 is a lower bound

Remark 6. The simplest type of a sequence is a monotone sequence. Checking that a monotone sequence converges is as easy as checking that it is bounded. It is also easy to find the limit for a convergent monotone sequence, provided we can find the supremum or infimum of a countable set of numbers.

2.2 Convergent and divergent sequences

2.2.1 Converges sequences

Definition 14. A sequence of real numbers (u_n) is called convergent if there is (exists) a real number l such that

$$\forall \varepsilon > 0,\ \exists n_\epsilon > 0,\ \forall n \in \mathbb{N},\ n \geq n_\epsilon \Rightarrow |u_n - l| < \varepsilon.$$

In this case, we say that the sequence (u_n) tends l, and we write $\lim\limits_{n\to+\infty} u_n = l$, ($l$ is finite).

Example 7. Let us show the sequence defined by the general term $u_n = \frac{n^2+1}{n^2+n}$ converges and $\lim\limits_{n\to+\infty} u_n = 1$.

In fact. Given $\epsilon > 0$, find $n_\epsilon > 0$ such that $n \geq n_\epsilon \Leftrightarrow \frac{1}{n} \leq \frac{1}{n_\epsilon}$, we have

$$\begin{aligned} |u_n - 1| &= \left|\frac{n^2+1}{n^2+n} - 1\right| \\ &= \left|\frac{1-n}{n^2+n}\right| \\ &= \frac{n-1}{n^2+n} \\ &\leq \frac{n}{n^2+n} = \frac{1}{n+1} \\ &\leq \frac{1}{n} \end{aligned}$$

$$\leq \frac{1}{n_\epsilon}$$
$$< \epsilon.$$

It's sufficient to take $\frac{1}{n_\epsilon} < \epsilon$. Therefore $\lim_{n\to+\infty} u_n = 1$.

Proposition 1. *A convergent sequence has a unique limit (The limit of a convergent sequence is unique).*

Definition 15. if the sequence is not converge, is called diverge .

Proposition 2. (i) *Every convergent sequence is bounded.*
(ii) *A monotone sequence is bounded if and only if it is convergent*

Proposition 3. *Every bounded monotone sequence of real numbers converges. That is,*
(i) *Every decreasing sequence and has a upper bouned has (admit) a limit (converge).*
(ii) *Every increasing sequence and has a lower bouned (admit) a limit (converge).*

Example 8. For each of the following sequences, use the Monotone Convergence Theorem to show the sequence converges and find its limit

1) $u_n = \frac{4^n}{n!}$, 2) $\begin{cases} v_{n+1} = \frac{v_n}{2} + \frac{1}{2v_n}, \ n \geq 2. \\ v_0 = 2. \end{cases}$.

1) Writing out the first few terms, we see that

$$u_n = \left\{4, 8, \frac{32}{3}, \frac{32}{3}, \frac{128}{15}, \dots\right\}.$$

At first, the terms increase. However, after the third term, the terms decrease. In fact, the terms decrease for all $n \geq 3$. We can show this as follows

$$u_{n+1} = \frac{4^{n+1}}{(n+1)!} = \frac{4}{n+1} \times \frac{4^n}{n!} = u_n \times \frac{4}{n+1} \leq u_n, \quad if\ n \geq 3.$$

Therefore, the sequence is decreasing for all $n \geq 3$. Further, the sequence is bounded below by because for all positive integers n. Therefore, by the monotone convergence theorem, the sequence (u_n) converges.
To find the limit, we use the fact that the sequence converges that is $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} u_{n+1} = L$. Combining this fact with the equation

$$u_{n+1} = u_n \times \frac{4}{n+1},$$

and taking the limit of both sides of the equation

$$L = \lim_{n\to+\infty} [u_{n+1}] = \lim_{n\to+\infty} \left[u_n \times \frac{4}{n+1}\right] = L \times 0 = 0.$$

2) Writing out the first several terms

$$v_n = \left\{2, \frac{5}{4}, \frac{41}{40}, \frac{3281}{3280}, \dots\right\},$$

we can conjecture that the sequence is decreasing and bounded below by 1. To show that the sequence is bounded below by 1, we can show that

$$\frac{v_n}{2} + \frac{1}{2v_n} \geq 1.$$

To show this, first rewrite

$$\frac{v_n}{2} + \frac{1}{2v_n} = \frac{v_n^2 + 1}{2v_n}.$$

Since $v_1 > 0$ and v_2 is defined as a sum of positive terms $v_2 > 0$, Similarly, all terms $v_n > 0$. Therefore,

$$\frac{v_n^2 + 1}{2v_n} \geq 1 \Longleftrightarrow v_n^2 + 1 \geq 2v_n.$$

Rewriting the inequality $v_n^2 + 1 \geq 2v_n$ as $v_n^2 - 2v_n + 1 \geq 0$ and using the fact that

$$v_n^2 - 2v_n + 1 = (v_n - 1)^2 \geq 0.$$

because the square of any real number is nonnegative, we can conclude that

$$v_n^2 - 2v_n + 1 = (v_n - 1)^2 \geq 0 \Leftrightarrow \frac{v_n}{2} + \frac{1}{2v_n} \geq 1.$$

To show that the sequence is decreasing, we must show that $v_{n+1} \leq v_n, \quad \forall n \geq 1$. Since $v_n^2 \geq 1$, it follows that

$$v_n^2 + 1 \leq 2v_n^2 \Rightarrow \frac{v_n}{2} + \frac{1}{2v_n} \leq v_n$$

$$\Rightarrow v_{n+1} \leq v_n, \ \forall n \geq 1.$$

Since (v_n) is bounded below and decreasing, by the monotone convergence theorem, it converges.
To find the limit, we use the fact that the sequence converges that is $\lim\limits_{n\to+\infty} v_n = \lim\limits_{n\to+\infty} v_{n+1} = L$. We have

$$v_{n+1} = \frac{v_n}{2} + \frac{1}{2v_n} \Leftrightarrow \lim_{n\to+\infty} v_{n+1} = \lim_{n\to+\infty} \frac{v_n}{2} + \frac{1}{2v_n}$$

$$\Leftrightarrow L = \frac{L}{2} + \frac{1}{2L}$$

$$\Leftrightarrow L^2 = 1$$

$$\Leftrightarrow L = \pm 1.$$

Since all the terms are positive, then $\lim\limits_{n\to+\infty} v_n = 1$

You should examine a few examples and convince yourself that then there is no limit (i.e., divergent sequence) or there is a limit (i.e., convergent sequence).

Proposition 4. *(Cauchy sequence or Cauchy's Convergence Criterion) A sequence real numbers converges if and only if satisfies thevfollowing Cauchy's convergence Criterion :*

$$\forall\varepsilon > 0,\ \exists n_\epsilon > 0,\ \forall p,\ q \in \mathbb{N},\ (P \geq n_\epsilon,\ q \geq n_\epsilon \Rightarrow |u_p - u_q| < \varepsilon)\,.$$

Proposition 5. *Let (u_n), (v_n) et (w_n) are three sequences satisfying :*
$(v_n) \leq (u_n) \leq (w_n)$ *from a certain rank. If* $\lim\limits_{n\to+\infty} v_n = \lim\limits_{n\to+\infty} w_n = l$, *then* $\lim\limits_{n\to+\infty} u_n = l$.

Proposition 6. *(Comparaison test) Let (u_n) and (v_n) two sequences satisfied:*
(i) $(v_n) \leq (u_n)$ *from a certain rank,*
(ii) If $\lim\limits_{n\to+\infty} v_n = +\infty$ *(respectively* $\lim\limits_{n\to+\infty} u_n = -\infty$*). Then,* $\lim\limits_{n\to+\infty} u_n = +\infty$ *(respectively* $\lim\limits_{n\to+\infty} v_n = -\infty$*).*

Exercise 4. Use the squeeze theorem to find the limit of each of the following sequences.

$$1)\ u_n = \frac{\cos(n)}{n^2}, \quad 2)\ v_n = \left(-\frac{1}{2}\right)^n.$$

Solution : 1) Since

$$-1 \leq \cos(n) \leq 1,\ \forall n \in \mathbb{N} \Rightarrow -\frac{1}{n^2} \leq \frac{\cos(n)}{n^2} \leq \frac{1}{n^2},\ \forall n \in \mathbb{N}^*,$$

and

$$\lim_{n\to+\infty} -\frac{1}{n^2} = \lim_{n\to+\infty} \frac{1}{n^2} = 0.$$

Then, according to the squeeze theorem, we conclude that $\lim\limits_{n\to+\infty} u_n = 0$.
2) Since

$$-\frac{1}{2^n} \leq \left(-\frac{1}{2}\right)^n \leq \frac{1}{2^n},$$

and

$$\lim_{n\to+\infty} -\frac{1}{2^2} = \lim_{n\to+\infty} \frac{1}{2^n} = 0.$$

Then, according to the squeeze theorem, we conclude that $\lim\limits_{n\to+\infty} v_n = 0$.

2.3 Subsequences

Definition 16. If f injective maps increasing from $\mathbb{N}$ into $\mathbb{N}$, then the sequence of general term $v_n = u_{f(n)}$ is called a subsequence of the sequence (u_n).

Example 9. 1) If $f : n \to 2n$ and $u_n = (-2)^n$, $\forall n \in \mathbb{N}$, then $v_n = u_{2n} = 4^n$, $\forall n \in \mathbb{N}$, is a subsequence of (u_n) .
2) If $f : n \to 2^n$ and $u_n = \frac{1}{n}$, $\forall n \in \mathbb{N}$, then $v_n = u_{2^n} = \frac{1}{2^n}$, $\forall n \in \mathbb{N}$, is a subsequence of (v_n).

Proposition 7. *If a sequence of real numbers (u_n) is converge to the limit l, then the subsequence (v_n) of (u_n) is also converge to l.*

2.4 Cauchy sequence

Definition 17. A sequence (u_n) is called Cauchy sequence if

$$\forall \varepsilon > 0,\ \exists n_\epsilon \in \mathbb{N} \,:\, \forall\,(p,q) \in \mathbb{N}^2\ (p \geq n_\epsilon,\ q \geq n_\epsilon) \Rightarrow |u_p - u_q| < \varepsilon.$$

Example 10. The sequence defined by the general term $u_n = \frac{1}{n},\ n > 0$ is a Cauchy sequence.

In fact. Given $\epsilon > 0$, find $n_\epsilon \in \mathbb{N}$. For $p \geq n_\epsilon$, $q \geq n_\epsilon$, we have

$$\left|\frac{1}{p} - \frac{1}{q}\right| \leq \left|\frac{1}{p}\right| + \left|\frac{1}{q}\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon.$$

It's sufficient to take $n_\epsilon > \frac{\epsilon}{2}$ such that $\frac{1}{p} < \frac{\epsilon}{2}$ and $\frac{1}{q} < \frac{\epsilon}{2}$.

Example 11. The sequence defined by the general term $u_n = \frac{n+1}{n},\ n > 0$ is a Cauchy sequence.

In fact. Given $\epsilon > 0$, find $n_\epsilon \in \mathbb{N}$. For $p \geq n_\epsilon$, $q \geq n_\epsilon$, we have

$$\begin{aligned}
\left|\frac{p+1}{p} - \frac{q+1}{q}\right| &\leq \left|\frac{(p+1)\,q - (q+1)\,p}{pq}\right| \\
&\leq \left|\frac{pq + q - qp - p}{pq}\right| \\
&\leq \left|\frac{q-p}{pq}\right| \\
&\leq \left|\frac{p}{pq}\right| + \left|\frac{-p}{pq}\right| \\
&\leq \left|\frac{p}{pq}\right| + \left|\frac{p}{pq}\right| \\
&\leq \left|\frac{1}{q}\right| + \left|\frac{1}{q}\right| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon.
\end{aligned}$$

It's sufficient to take $n_\epsilon > \frac{\epsilon}{2}$ such that $\frac{1}{p} < \frac{\epsilon}{2}$ and $\frac{1}{q} < \frac{\epsilon}{2}$.

Proposition 8. *A Cauchy sequence is bounded.*

Theorem 4. *A sequence of real numbers is Cauchy if and only if it converges.*

2.5 Adjacents sequences

Definition 18. Two numerical sequences(v_n) and (u_n) are said to be adjacent if one is increasing, the other is decreasing and $\lim\limits_{n\to+\infty}[u_n - v_n] = 0$.

Proposition 9. *If (v_n) and (u_n) are adjacent. Then, the two sequences are adjacent and converge towards the same limit.*

Exercise 5. Applying the definition of the limit of a sequence to prove that the following sequences converge to the given limit (towards the indicated limit) :

1) $u_n = \frac{4n-1}{2n+1},\ n \geq 1,\ l = 2.$

2) $u_n = \sum\limits_{k=0}^{n} q^n,\ q \neq 1, \quad l = \frac{1}{1-q}$ *and* $\ln|q| < 0.$

3) $u_n = 2^{(n)^{\frac{1}{2}}},\ n \in \mathbb{N}\, and\, l = +\infty.$

Solution: 1) Let $\epsilon > 0$, we have

$$|u_n - 2| < \epsilon \Longleftrightarrow \left|\frac{4n-1}{2n+1} - 2\right| < \epsilon$$

$$\Longleftrightarrow \left|\frac{-3}{2n+1}\right| < \epsilon$$

$$\Longleftrightarrow \frac{3}{2n+1} < \epsilon.$$

This last inequality is satisfied for all $n \geq \frac{1}{2}\left(\frac{3}{\epsilon} - 1\right)$. We set $n_\epsilon = \left[\frac{1}{2}\left(\frac{3}{\epsilon} - 1\right)\right] + 1$, where $\left[\frac{1}{2}\left(\frac{3}{\epsilon} - 1\right)\right]$ denoted the integer part of $\frac{1}{2}\left(\frac{3}{\epsilon} - 1\right)$.
We deduce that

$$(\forall n \in \mathbb{N})\, n \geq n_\epsilon \Longrightarrow \left|\frac{4n-1}{2n+1} - 2\right| < \epsilon.$$

Thus $\lim\limits_{n\to+\infty} u_n = 2$.

2) It is well known that $\sum\limits_{k=0}^{n} q^n = \frac{1-q^{n+1}}{1-q},\ q \neq 1, \quad l = \frac{1}{1-q}$ *et* $\ln|q| < 0$. Let $\epsilon > 0$, we have

$$\left|u_n - \frac{1}{1-q}\right| < \epsilon \Longleftrightarrow \left|\frac{1-q^{n+1}}{1-q} - \frac{1}{1-q}\right| < \epsilon$$

$$\Longleftrightarrow \left|\frac{q^{n+1}}{1-q}\right| < \epsilon$$

$$\Longleftrightarrow \frac{|q^{n+1}|}{|1-q|} < \epsilon$$

$$\Longleftrightarrow \left|q^{n+1}\right| < \epsilon\,|1-q|$$

$$\Longleftrightarrow n+1 > \frac{\ln(\epsilon\,|1-q|)}{\ln|q|}.$$

It sufficient to take $n > \left[\frac{ln(\epsilon|1-q|)}{ln|q|} - 1\right] + 1$, where $\left[\frac{ln(\epsilon|1-q|)}{ln|q|} - 1\right]$ denoted the integer part of $\frac{\ln(\epsilon|1-q|)}{\ln|q|} - 1$. $\lim\limits_{n\to+\infty} u_n = \frac{1}{1-q}$.

3) $u_n = 2^{(n)^{\frac{1}{2}}}$, $n \in \mathbb{N}\; and\; l = +\infty$. Let A be a positive strictly real number. We have

$$u_n = 2^{(n)^{\frac{1}{2}}} > A \Rightarrow n^{\frac{1}{2}} > \frac{\ln(A)}{\ln(2)}$$

$$\Rightarrow n > \left(\frac{\ln(A)}{\ln(2)}\right)^2.$$

It sufficient to take $A > 0$, $n_\epsilon = \left[\left(\frac{\ln(A)}{\ln(2)}\right)^2\right] + 1$. Then, $\lim\limits_{n\to+\infty} u_n = +\infty$.

Exercise 6. For each of the following sequences, determine whether or not the sequence converges. If it converges, find its limit.

$$1)\ u_n = 5 - \frac{3}{n^2},\ n \in \mathbb{N}^*\ 2)\ v_n = \frac{3n^4 - 7n^2 + 5}{6 - 4n^2},\ n \in \mathbb{N},$$

$$3)\ w_n = \frac{2^n}{n^2},\ n \in \mathbb{N}^*,\ 4)\quad t_n = \left(1 + \frac{4}{n}\right)^n,\ n \in \mathbb{N}^*.$$

$$5)\ l_n = \frac{2^n + 3^n}{4^n},\ n \in \mathbb{N},\ 6)\quad x_n = \frac{(n!)^2}{(2n)!},\ n \in \mathbb{N}^*.$$

Solution : 1) Since $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} \left(5 - \frac{3}{n^2}\right) = 5$, then, the sequence (u_n) converges and its limit is 5.

2) By factoring out of the numerator and denominator and using the limit laws above, we have

$$\lim_{n\to+\infty} v_n = \lim_{n\to+\infty} \left(\frac{3n^4 - 7n^2 + 5}{6 - 4n^2}\right)$$

$$= \lim_{n\to+\infty} \frac{n^2\left(3n^2 - 7 + \frac{5}{n^2}\right)}{n^2\left(\frac{6}{n^2} - 4\right)}$$

$$= \lim_{n\to+\infty} \frac{3n^2 - 7 + \frac{5}{n^2}}{\frac{6}{n^2} - 4} = -\frac{3}{4}.$$

So, the sequence (v_n) converges and its limit is $-\frac{3}{4}$.

3) By applying L'Hôpital's rule two times succesively for the continuous function $f(x) = \frac{2^x}{x^2}$, we have

$$\lim_{n\to+\infty} f(x) = \lim_{n\to+\infty} \frac{2^x}{x^2}$$

$$= \lim_{n\to+\infty} \frac{2^x \ln(2)}{2x}$$

$$= \lim_{n\to+\infty} \frac{2^x \ln^2(2)}{2} = +\infty.$$

We conclude that the sequence (t_n) diverges.
4) We know that

$$\lim_{n\to+\infty} w_n = \lim_{n\to+\infty}\left(1+\frac{4}{n}\right)^n = e^4.$$

we conclude that the sequence (w_n) converges to e^4.
5) We know that, for all $n \in \mathbb{N}$, we have

$$2^n + 3^n \leq 3^n + 3^n = 2\times 3^n \Rightarrow \frac{2^n+3^n}{4^n} \leq \frac{2\times 3^n}{4^n}$$
$$\Rightarrow \frac{2^n+3^n}{4^n} \leq \frac{2\times 3^n}{4^n} = 2\left(\frac{3}{4}\right)^n.$$

So, since

$$\lim_{n\to+\infty} 2\left(\frac{3}{4}\right)^n = 0, \quad \textit{because} \quad -1 \leq \frac{3}{4} \leq 1,$$

then,

$$\lim_{n\to+\infty} l_n = \lim_{n\to+\infty}\frac{2^n+3^n}{4^n} = 0,$$

we conclude that the sequence (l_n) converges to 0.
6) We know that, for all $n \in \mathbb{N}$, we have

$$\frac{x_{n+1}}{x_n} = \frac{n!}{(n+1)(n+2)\dots(2n)} = \frac{1\times 2\times 3\times\cdots\times n}{(n+1)(n+2)\dots(2n)} \leq \left(\frac{1}{2}\right)^n.$$

So, since $\lim\limits_{n\to+\infty}\left(\frac{1}{2}\right)^n = 0$, then $\lim\limits_{n\to+\infty} x_n = 0$, we conclude that the sequence (x_n) converges to 0.

Exercise 7. Among the following sequences, show those that are bounded (For each of the following sequences, show whether or not the sequence bounded).

1) $u_n = \frac{(-1)^n n+25}{\sqrt{n^2+4}}$, 2) $u_n = \sum\limits_{k=1}^{n}\frac{1}{k+n}$.

Solution: 1) On a

$$\begin{aligned}|u_n| = \left|\frac{(-1)^n n+25}{\sqrt{n^2+4}}\right| &\leq \frac{|(-1)^n n|+25}{\sqrt{n^2+4}}\\ &= \frac{n+25}{\sqrt{n^2+4}}\\ &\leq \frac{n}{\sqrt{n^2}}+\frac{25}{\sqrt{4}}\\ &\leq 1+\frac{25}{2} = \frac{27}{2},\end{aligned}$$

then, (u_n) is bounded.

2)

$$
\begin{aligned}
0 \leq u_n = \sum_{k=1}^{n} \frac{1}{k+n} &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \\
&\leq \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1} \\
&= \frac{n}{1+n} < 1,
\end{aligned}
$$

then, (u_n) is bounded.

Exercise 8. Study the following monotony and possibly deduce their nature :

1) $u_n = \sum\limits_{k=1}^{n} \frac{1}{k+n}$

Solution: First, we have

$$
\begin{aligned}
u_{n+1} - u_n = &\left(\frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{(n+1)+n} + \frac{1}{2(n+1)} \right) \\
&- \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n-1} + \frac{1}{2n} \right) \\
= &\frac{1}{2n+1} + \frac{1}{2(n+1)} - \frac{1}{n+1} \\
= &\frac{1}{2n+1} - \frac{1}{2(n+1)} \\
= &\frac{1}{2(2n+1)(n+1)} > 0,
\end{aligned}
$$

then, the sequence (u_n) is strictly decreasing.
On the other hand,

$$
\begin{aligned}
u_n = \sum_{k=1}^{n} \frac{1}{k+n} &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \\
&\leq \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1} \\
&= \frac{n}{1+n} < 1,
\end{aligned}
$$

then, (u_n) bounded above.
So, since the sequence (u_n) is strictly decreasing and (u_n) is bounded above, then it's converges.

Exercise 9. Let the sequence defined by $u_0$is a given real number and $u_{n+1} = (u_n)^2 - 2, \quad \forall n \in \mathbb{N}$.

1) If $u_0 = \sqrt{2}$, calculate u_1, u_2 and u_3.
2) Determine u_4. Deduce the nature of the sequence (u_n).
3) Determine the nature of the sequence (u_n) if $u_0 = 2$.
4) Find another value u_0 for which the sequence (u_n) is constant.

Solution: 1) If $u_0 = \sqrt{2}$, we have $u_1 = 0$, $u_2 = -2$ and $u_3 = 2$.

2) We have $u_4 = u_3^2 - 2 = 2$. Further, we have $u_n = 2$, $\forall n \geq 3$. Thus, the sequence (u_n) converges to 2.

3) If $u_0 = 2$, (u_n) is constant

4) The sequence (u_n) will be constant if

$$\begin{aligned} u_0 = u_0^2 - 2 &\Longleftrightarrow u_0^2 - u_0 - 2 = 0 \\ &\Longleftrightarrow (u_0 - 2)(u_0 + 1) = 0 \\ &\Longleftrightarrow u_0 = -1 \; ou \; u_0 = 2. \end{aligned}$$

Exercise 10. On considère deux suites (u_n) et (v_n) définies par

$$\begin{cases} u_n = \sum\limits_{k=0}^{n} \frac{1}{k!}, \; \forall n \in \mathbb{N}, \\ u_0 = 1, \end{cases}$$

et

$$\begin{cases} v_n = u_n + \frac{1}{n!}, \; \forall n \in \mathbb{N}, \\ v_0 = 1. \end{cases}$$

1) Show that (u_n) and (v_n) are adjacent.
2) Can their limit be a rational number.

Solution: 1) It is well known that

$$\forall n \in \mathbb{N}, \; u_n \leq v_n.$$

Further

$$\lim_{n \to +\infty} (v_n - u_n) = \lim_{n \to +\infty} \frac{1}{n!} = 0.$$

Also, since $u_{n+1} - u_n = \frac{1}{(n+1)!} = 0$, $\forall n \in \mathbb{N}$ (respectively $v_{n+1} - v_n = \frac{1-n}{(n+1)!} = 0$, $\forall n > 1$) , then, the sequence (u_n) is strictly decreasing (respectively (v_n) is strictly increasing).
So the two sequences are adjacent and converge towards the same limit l satisfied $u_n < l < v_n$, $\forall n \in \mathbb{N}$.
2) We will demonstrate that the limit l can not be a rational number. Reasoning by the absurd. Suppose that l is a rational number, i.e.,

$$\begin{aligned} l = \frac{p}{q}, \quad p, \; q \in \mathbb{N}^* &\Longrightarrow u_q < \frac{p}{q} < v_q \\ &\Longrightarrow q! u_q < p(q-1)! < q! v_q \\ &\Longrightarrow q! u_q < p(q-1)! < q! u_q + 1. \end{aligned}$$

Thus the integer $p(q-1)!$ is between two consecutive integers, hence the contradiction. So, the limit l is irrational.

Exercise 11. Show that the sequence (u_n) defined by $u_n = \cos\left(\frac{1}{n}\right)$ is Cauchy sequence.

Solution: We will demonstrate that $\forall \varepsilon > 0$, $\exists n_\epsilon \in \mathbb{N} : \forall (p, q) \in \mathbb{N}^2$ $(p \geq n_\epsilon,\, q \geq n_\epsilon) \Rightarrow |u_p - u_q| < \varepsilon$.
Let $\epsilon > 0$, then, according to the trigonometric transformations, we have

$$|u_p - u_q| = \left|\cos\left(\frac{1}{p}\right) - \cos\left(\frac{1}{q}\right)\right| \leq 2\left|\sin\left(\frac{1}{2p} + \frac{1}{2q}\right)\right| \leq \frac{1}{p} + \frac{1}{q} \leq 2\frac{1}{n_\epsilon} < \epsilon,$$

because if x is small, $x \geq 0$, $|\sin(x)| \leq x$. Thus, if p, $q \geq n_\epsilon$, then it sufficient that (it is enough that) $\frac{2}{n_\epsilon} < \epsilon$ i.e., $n_\epsilon > \frac{2}{\epsilon}$.

Chapter 3

Limits

In this section we study limits of real-valued functions of a real variable. Before we come to definitions, let us start with a little notation for limits.

3.1 Notion of the limit in a point

Let $f : I \to \mathbb{R}$, I is an open interval of $\mathbb{R}$.

Definition 19. We say that f admit (has) a finite limit l as x approaches $x_0 \in I$ and denoted by $\lim\limits_{x \to x_0} f(x) = l$ if:

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in I,\ |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon.$$

Remark 7. $\lim\limits_{x \to x_0} f(x) = l$ which should be read as : The limit of $f(x)$ as x approaches (tends) x_0 is l. We can also write the above limit as

$$f(x) \to l\; as\; x \to x_0.$$

3.2 Left- and right-hand limits (Informal definition of one-sided limits)

Let x_0 is a left extremity of interval $J \subset I$ and $x_0 \notin J$.

Definition 20. If $\lim\limits_{x \in J,\, x \to x_0} f(x) = l$, we say that f admit (has) a right-hand-limit l at x_0 denoted by $\lim\limits_{x \to x_0^+} f(x) = l$.

Remark 8. (i) $x \to x_0^+$ that is the x-values are always less than x_0, we say that x approaches x_0 from below (the x-values lie to the left of x_0 on a sketch of the graph).

$(ii) x \to x_0^+$ that is the x-values are always greater than x_0, we say that x approaches x_0 from below (the x-values lie to the left of x_0 on a sketch of the graph).

Let x_0 is a right extremity of interval $J \subset I$ and $x_0 \notin J$.

Definition 21. If $\lim\limits_{x\in J,\, x\to x_0} f(x) = l$, we say that f admit (has) a right-hand-limit l at x_0 denoted by $\lim\limits_{x\to x_0^-} f(x) = l$. That is if the left-hand and right-hand limits as x approaches x_0 exist and are equal, then the limit as x approaches x_0 exists and is equal to the one-sided limits.

Theorem 5. *$\lim\limits_{x\to x_0} f(x) = l$ if and only if $\lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^-} f(x) = l$.*

Remark 9. (i) If they both the left-hand and right-hand limits exist but are different, then the limit as x approaches x_0 does not exist.

3.3 Extension of the concept of the limit

a) Assume that I unbounded and $l \in \mathbb{R}$.

1) $\lim\limits_{x\to+\infty} f(x) = l$ (respectively $\lim\limits_{x\to-\infty} f(x) = l$) signify (that is)

$$\forall\varepsilon > 0,\ \exists A > 0,\ \forall x \in I,\ x > A \Rightarrow |f(x) - l| < \varepsilon,$$

respectively

$$\forall\varepsilon > 0,\ \exists A > 0,\ \forall x \in I,\ x < -A \Rightarrow |f(x) - l| < \varepsilon.$$

2) $\lim\limits_{x\to+\infty} f(x) = +\infty$, ($f(x)$ becomes arbitrarily large) (respectively $\lim\limits_{x\to-\infty} f(x) = +\infty$) signify (that is)

$$\forall B > 0,\ \exists A > 0,\ \forall x \in I,\ x > A \Rightarrow f(x) > B,$$

respectively

$$\forall B > 0,\ \exists A > 0,\ \forall x \in I,\ x < -A \Rightarrow f(x) > B.$$

3) $\lim\limits_{x\to+\infty} f(x) = -\infty$, (becomes arbitrarily large) (respectively $\lim\limits_{x\to-\infty} f(x) = -\infty$) signify

$$\forall B > 0,\ \exists A > 0,\ \forall x \in I,\ x > A \Rightarrow f(x) < -B,$$

respectively

$$\forall B > 0,\ \exists A > 0,\ \forall x \in I,\ x < -A \Rightarrow f(x) < -B.$$

b) Suppose that I bounded and $x_0 \in I$.
$\lim\limits_{x\to x_0} f(x) = +\infty$ ($\lim\limits_{x\to x_0} f(x) = -\infty$ respectively) signify

$$\forall B > 0,\ \exists\delta > 0,\ \forall x \in I,\ |x - x_0| < \delta \Rightarrow f(x) > B,$$

respectively

$$\forall B > 0,\ \exists\delta > 0,\ \forall x \in I,\ |x - x_0| < \delta \Rightarrow f(x) < -B.$$

3.4 General theoremes of the limits (Useful theorem about limits)

Theorem 6. *Let $I \subset \mathbb{R}$, $f : I \to \mathbb{R}$ is a function, l is a number (finite or infinite) and $x_0 \in I$. Then*

$\lim\limits_{x \to x_0} f(x) = l$, $l \in \mathbb{R}$ *if and only if for any* $(u_n) \in I$, $\lim\limits_{x \to +\infty} u_n = x_0$ *the sequence* $(f(u_n))$ *tends to* l.

Theorem 7. *Let f, g two functions, $I \subset \mathbb{R}$ and $x_0 \in I$ ($x_0 = +\infty$ or $x_0 = -\infty$).*

If $\lim\limits_{x \to x_0} f(x) = l$ *and* $\lim\limits_{x \to x_0} g(x) = l'$, *then*

1) $\lim\limits_{x \to x_0} [f(x) + g(x)] = l + l'$ *(exept if $l = +\infty$, $l' = -\infty$).*

2) $\lim\limits_{x \to x_0} [(f \cdot g)(x)] = l \cdot l'$ *(exept if $l = 0$, $l' = \pm\infty$).*

3) $\lim\limits_{x \to x_0} [(\lambda f)(x)] = \lambda l, \quad \forall \lambda \in \mathbb{R}$.

Theorem 8. *Squeeze theorem (or sandwich theorem or pinch theorem)).*
Let $a \in \mathbb{R}$ and let f, g, h be three functions so that

$$f(x) \leq g(x) \leq h(x),$$

for all x in an interval around a, except possibly exactly at $x = a$. Then if

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l,$$

then

$$\lim_{x \to a} g(x) = l.$$

Example 12. Compute the limit

$$\lim_{x \to 0} x^2 \sin\left(\frac{\pi}{x}\right).$$

Since $-1 \leq \sin(x) \leq 1,\ \forall x \in \mathbb{R}$, we have

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1,\ \forall x \in \mathbb{R}^*.$$

Multiplying the above by x^2 we have

$$-x^2 \leq x^2 \sin\left(\frac{\pi}{x}\right) \leq x^2,\ \forall x \in \mathbb{R}^*.$$

Since

$$\lim_{x \to 0} x^2 = \lim_{x \to 0} -x^2 = 0,$$

by the sandwich (or squeeze or pinch) theorem, we have

$$\lim_{x \to 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0.$$

Proposition 10. *Let f, g be two functions defined in interval $I =]a, b[$ (a and b are finites) and $x_0 \in I$ (x_0 finite, infinite or extremity of I). If g is bounded and $\lim\limits_{x \to x_0} f(x) = 0$, then*

$$\lim_{x \to x_0} g(x) \cdot f(x) = 0.$$

Example 13. We will calculate $\lim\limits_{x \to 0} [x \sin(x)]$. We put

$$\begin{cases} f(x) = x, \\ g(x) = \sin(x). \end{cases}$$

Since $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x = 0$ and $g : x \mapsto \sin(x)$ is bounded on $\mathbb{R}$ particulary near 0, then from Proposition 10, we have

$$\lim_{x \to 0} [x \sin(x)] = 0.$$

Example 14. We will calculate $\lim\limits_{x \to +\infty} \frac{\sin(x)}{x} = \lim\limits_{x \to +\infty} \frac{1}{x} \cdot \sin(x)$. We put

$$\begin{cases} f(x) = \frac{1}{x}, \\ g(x) = \sin(x). \end{cases}$$

Since $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \frac{1}{x} = 0$ and $g : x \mapsto \sin(x)$ is bounded on $\mathbb{R}$ particulary near $+\infty$, then from Proposition 10, we have

$$\lim_{x \to +\infty} \frac{\sin(x)}{x} = 0.$$

3.5 Undetermined forms

Undetermined forms are :

$$\infty \times 0, \ \frac{\infty}{\infty}, \ \frac{0}{0}, \ \ +\infty + (-\infty), \ 0^0, \ 1^\infty, \ \infty^0.$$

3.6 Several limits (remarkable limits)

1) $\lim\limits_{x \to 0} \frac{\sin(x)}{x} = 1$ *and* $\lim\limits_{x \to 0} \frac{\tan(x)}{x} = 1$.
2) $\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$ *and* $\lim\limits_{x \to +\infty} \left(1 + \frac{\lambda}{x}\right)^x = e^\lambda, \ \lambda \in \mathbb{R}$.
3) $\lim\limits_{x \to +\infty} \frac{\ln(x)}{x} = 0^+$ *and* $\lim\limits_{x \to +\infty} \frac{\ln(x)}{x^p} = +\infty, \ p \in \mathbb{Q}_+^*$.
4) $\lim\limits_{x \to 0^+} x \ln(1 + x) = 0^-$ *and* $\lim\limits_{x \to 0^+} x^p \ln(x), \ p \in \mathbb{Q}_+^*$.
5) $\lim\limits_{x \to 0} \frac{\ln(1+x)}{x} = 1$.
6) $\lim\limits_{x \to +\infty} \frac{e^x}{x} = 0$ *and* $\lim\limits_{x \to +\infty} \frac{e^x}{x^p} = 0, \ p \in \mathbb{Q}_+^*$.
7) $\lim\limits_{x \to -\infty} x e^x = 0$ *and* $\lim\limits_{x \to -\infty} x^p e^x = 0, \ p \in \mathbb{Q}_+^*$.
8) $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$.

Example 15. To arrive at the definition of limit, I want to start with a very simple examples.

1)

$$\lim_{x\to+\infty}\frac{x-\ln(x)}{x+\ln(x)}=\lim_{x\to+\infty}\frac{x\left[1-\frac{\ln(x)}{x}\right]}{x\left[1+\frac{\ln(x)}{x}\right]}=\lim_{x\to+\infty}\frac{1-\frac{\ln(x)}{x}}{1+\frac{\ln(x)}{x}}=1.$$

2)

$$\lim_{x\to0}\frac{\ln(1+4x)}{16x}=\frac{1}{4}\lim_{x\to0}\frac{\ln(1+4x)}{4x}=\frac{1}{4}.$$

3)

$$\lim_{x\to0}\frac{e^{x^2}-1}{5x}=\lim_{x\to0}\frac{x^2}{5x}\cdot\frac{e^{x^2}-1}{x^2}=0\times1=0.$$

4)

$$\lim_{x\to+\infty}\left(1+\frac{2}{x}\right)^x=\lim_{x\to+\infty}e^{x\ln\left(1+\frac{2}{x}\right)}=\lim_{x\to+\infty}e^{2\frac{\ln\left(1+\frac{2}{x}\right)}{\frac{2}{x}}}=e^2.$$

5)

$$\lim_{x\to+\infty}\frac{\ln^2(x)}{x}=\lim_{x\to+\infty}\left(\frac{\ln(x)}{\sqrt{x}}\right)^2=0.$$

6)

$$\lim_{x\to0^+}\sqrt{x}\ln(x)=0^-.$$

7)

$$\lim_{x\to+\infty}(x+2)e^{-x}=\lim_{x\to+\infty}\left[xe^{-x}+2e^{-x}\right]=\lim_{x\to+\infty}\left[-\left(-xe^{-x}\right)+2e^{-x}\right]=0.$$

Exercise 12. Calculate (Find) the following limits :

1) $\lim\limits_{x\to1^+}(2x-2)\ln\sqrt{4x^2-4}$.
2) $\lim\limits_{x\to0}\frac{1}{x}\ln\frac{2+x}{2-x}$.
3) $\lim\limits_{x\to+\infty}\frac{e^{2x}+e^x+1}{xe^x}$.
4) $\lim\limits_{x\to0}\frac{1}{x^2}e^{-\frac{1}{x}}$.
5) $\lim\limits_{x\to0}\frac{e^x-e^{4x}}{x}$.
6) $\lim\limits_{x\to0}\frac{e^{x^2}-1}{\tan^2(x)}$.
7) $\lim\limits_{x\to+\infty}\left(\frac{x-1}{x+1}\right)^x$.

Solution: We will appear the new expression of function such that the undetermined forms is lifted and the remarkable limits will be used to caculate the given limits

1) For all $x\in]1,+\infty[$, we have

$$\lim_{x\to1^+}(2x-2)\ln\sqrt{4x^2-4}=\lim_{x\to1^+}\frac{1}{2}(2x-2)\ln\left[4\left(x^2-1\right)\right]$$

$$=\lim_{x\to1^+}\left[(x-1)\ln4+(x-1)\ln\left(x^2-1\right)\right]$$

$$= \lim_{x\to 1^+} \left[(x-1)\ln 4 + (x-1)\ln(x-1) + (x-1)\ln(x+1)\right] = 0.$$

2)

$$\begin{aligned}
\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{2+x}{2-x}\right) &= \lim_{x\to 0}\frac{1}{x}\left[\ln(2+x) - \ln(2-x)\right] \\
&= \lim_{x\to 0}\frac{1}{x}\left[\ln\left[2\left(1+\frac{x}{2}\right)\right] - \ln\left[2\left(1-\frac{x}{2}\right)\right]\right] \\
&= \lim_{x\to 0}\frac{\ln\left(1+\frac{x}{2}\right)}{x} - \frac{\ln\left(1-\frac{x}{2}\right)}{x} \\
&= \frac{1}{2}\lim_{x\to 0}\frac{\ln\left(1+\frac{x}{2}\right)}{\frac{x}{2}} - \frac{\ln\left(1-\frac{x}{2}\right)}{\frac{x}{2}} \\
&= \frac{1}{2}\lim_{x\to 0}\frac{\ln\left(1+\frac{x}{2}\right)}{\frac{x}{2}} + \frac{\ln\left(1-\frac{x}{2}\right)}{-\frac{x}{2}} = \frac{1}{2}(1+1) = 1.
\end{aligned}$$

3)

$$\lim_{x\to+\infty}\frac{e^{2x}+e^{x}+1}{xe^{x}} = \lim_{x\to+\infty}\frac{e^{2x}\left(1+e^{-x}+e^{-2x}\right)}{xe^{x}} = \lim_{x\to+\infty}\frac{e^{x}}{x}\left(1+e^{-x}+e^{-2x}\right) = +\infty.$$

4)

$$\lim_{x\to 0^-}\frac{1}{x^2}e^{-\frac{1}{x}}. = \lim_{x\to 0^-}\left(\frac{1}{x}\right)^2 e^{-\frac{1}{x}} = \lim_{x\to 0^-}\frac{1}{\frac{e^{\frac{1}{x}}}{\left(\frac{1}{x}\right)^2}} = 0.$$

5)

$$\lim_{x\to 0}\frac{e^{x}-e^{4x}}{x} = \lim_{x\to 0}\frac{-e^{x}\left(e^{3x}-1\right)}{x} = \lim_{x\to 0} -3e^{x}\times\frac{e^{3x}-1}{3x} = -3\times 1 = -3.$$

6)

$$\lim_{x\to 0}\frac{e^{x^2}-1}{\tan^2(x)} = \lim_{x\to 0}\frac{e^{x^2}-1}{x^2}\times\frac{x^2}{\tan^2(x)} = \lim_{x\to 0}\frac{e^{x^2}-1}{x^2}\times\frac{1}{\left(\frac{\tan(x)}{x}\right)^2} = 1\times 1 = 1.$$

7)

$$\begin{aligned}
\lim_{x\to+\infty}\left(\frac{x-1}{x+1}\right)^{x} &= \lim_{x\to+\infty}\left(\frac{x+1-2}{x+1}\right)^{x} \\
&= \lim_{x\to+\infty}\left(1+\frac{-2}{x+1}\right)^{x} \\
&= \lim_{x\to+\infty}\left(1-\frac{2}{x+1}\right)^{x+1-1} \\
&= \lim_{x\to+\infty}\left(1-\frac{2}{x+1}\right)^{x+1}\cdot\left(1+\frac{-2}{x+1}\right)^{-1} = e^{-2}\times 1 = e^{-2}.
\end{aligned}$$

3.7 Continuous functions

A rule f that assigns to each member of a nonempty set D a unique member of a set F is a function from D to F . We write the relationship between a member x of D and the member y of F that f assigns to x as

$$y = f(x).$$

The set D is the domain of f, denoted by D_f. The members of F are the possible values of f. If $y_0 \in F$ and there is an x_0 in D such that $y_0 = f((x_0))$ then we say that f attains or assumes the value y_0. The set of values attained by f is the range of f. A real-valued function of a real variable is a function whose domain and range are both subsets of the reals. Although we are concerned only with real-valued functions of a real variable in this section, our definitions are not restricted to this situation.

3.7.1 Continuity at a point and over an interval

Definition 22. Let I be an open interval of $\mathbb{R}$. A function $f : I \to \mathbb{R}$ defined I is said continuous at point $x_0 \in I$, if $\lim\limits_{x \to x_0} f(x) = f(x_0)$, that is

$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in I,\ |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \epsilon.$

Definition 23. We say that f is continuous on (in) I if f is continuous at any point of I.

Example 16. The function $f : I =]0, +\infty[\to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sqrt{x-1}, & if\ x \geq 2, \\ \frac{4}{x^2}, & if\ 0 < x < 2, \end{cases}$$

is continuous in all point of I.

3.7.2 Left and right continuity at a point

Definition 24. We say that the function $f : I \to \mathbb{R}$ (I an open interval) is continuous from the right at (respectively from the left) at $x_0 \in I$, si $\lim\limits_{x \to x_0^+} f(x) = f(x_0)$, (respectively if $\lim\limits_{x \to x_0^-} f(x) = f(x_0)$).

Remark 10. f is continuous at point $x_0 \in I$ if and only if f is continuous from the left and from the right at $x_0 \in I$ (that is, $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0^-} f(x) = f(x_0)$).

Example 17. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = |x| = \begin{cases} x, & if\ x \geq 0, \\ -x, & if\ x < 0, \end{cases}$$

is is continuous from the right and form the left at each $x_0 \in \mathbb{R}$ and $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0^-} f(x)$, then its continuous at any point x_0.

Example 18. Let f be defined on $[0,2]$ by

$$f(x) = |x| = \begin{cases} x^2, & if\ 0 \leq x < 1, \\ x^2+1, & if\ 1 \leq x \leq 2. \end{cases}$$

We have $\lim\limits_{x \to 1^-} f(x) = 0 \neq 1 = f(1)$. Therefore, the function f is continuous from the right at 1, but not continuous at 1.

3.8 Fundamental properties of continuous functions on an interval

All continuous numerical function (function of a real numbers i.e, real valued function) on a bounded closed interval $[a,b]$ is

1) Bounded in $[a,b]$ that is, $\exists M,\ m \in [a,b],\ \forall x \in [a,b],\ m \leq f(x) \leq M$.

2) Reaches (attaind) its lower bound $m = \inf\limits_{x \in [a,b]} f(x)$ and its upper bound $M = \sup\limits_{x \in [a,b]} f(x)$ that is $\exists x_1,\ x_2 \in [a,b]$ such that $m = f(x_1)\ et\ M = f(x_2)$.

3) Reaches at least one time each value strictly between its minimum m and its maximum M, that is $\forall y \in]m, M[,\ \exists c \in [a,b]$ such that $y = f(x)$.

Conséquences: 1) If f is continuous on $[a,b]$ then, $f([a,b]) = [m, M]$.

2) The range of the interval by a continuous function is an inteval.

Example 19. 1) The function $x \longmapsto \frac{1}{1-x}$ on $]1,2]$ is continuous but not bounded because $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \frac{1}{1-x} = -\infty$.

2) The function $x \longmapsto x^3$ on $\mathbb{R} =]-\infty, +\infty[$ is continuous but not bounded because $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} x^3 = \pm\infty$.

3) The function $x \longmapsto \tan(x)(x)$ on $\left]-\frac{\pi}{2}, +\frac{\pi}{2}\right[$ is continuous but not bounded because $\lim\limits_{x \to \pm\frac{\pi}{2}} f(x) = \lim\limits_{x \to \pm\frac{\pi}{2}} \tan(x) = \pm\infty$.

3.9 Operations of continuous functions (Useful theorem about continuity)

Theorem 9. *If f and g are two functions from I in $\mathbb{R}$, continuous at a point $x_0 \in I$. Then, $f+g,\ f \cdot g,\ |f|$, and $\forall \lambda \in \mathbb{R}$, $(\lambda \cdot f)$ are a continuous fonctions at x_0. Further, if $g(x_0) \neq 0$ then, the function $\frac{f}{g}$ defined in $\{x \in I,\ g(x) \neq 0\}$ is continuous at x_0.*

3.10 Uniform continuity

Definition 25. Let $I \subset \mathbb{R}$ (I an open interval). We say that the function $f : I \to \mathbb{R}$ is uniformly continuous on I if :

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x,\ x' \in I,\ |x - x'| < \delta \Longrightarrow |f(x) - f(x')| < \epsilon,$$

where we emphasize that δ depends only on ϵ.

Example 20. The function $f(x) = 2x$ is uniformly continuous on $\mathbb{R}$. Since, let $\epsilon > 0,\ x, x' \in \mathbb{R}$, we have

$$|f(x) - f(x')| = 2|x - x'| < \epsilon \Rightarrow |x - x'| < \frac{\epsilon}{2}.$$

It sufficcient to choose $\delta = \frac{\epsilon}{2}$. Therefore, there exists $\delta = \frac{\epsilon}{2}$.

Theorem 10. *If $f : I \to \mathbb{R}$ is uniformly continuous on I then, its continuous on I.*

Remark 11. The inversely is false. For example, the function $f : x \to \frac{1}{x}$ continuous on $]0, 1]$, but not uniformely continuous since , $\forall \delta > 0$, the difference

$$\frac{1}{x} - \frac{1}{x+\delta} = \frac{\delta}{x(x+\delta)},$$

can be take enought values when x is sufficient near zéro.

Theorem 11. *(Heine theorem) f I is finite closed interval. All function $f : I \to \mathbb{R}$ is uniformely continuous on I.*

3.11 Continuity extension

Definition 26. Let $f : I \to \mathbb{R},\ x_0$ and l are reals numbers with $x_0 \notin I$. Suppose that

$$\lim_{x \to x_0 (x \neq x_0)} f(x) = l.$$

Then the function $g : I \cup \{x_0\} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} f(x), & si\ x \in I,\ (x \neq x_0), \\ l, & si\ x = x_0, \end{cases}$$

is continuous in x_0. The function g is said to be continuity extension of f in point x_0. The function g will denoted by $\widetilde{f}$.

Exercise 13. Study the continuity of the following functions on they definitions domain
1) $f(x) = x \sin\left(\frac{1}{x}\right)$.
2) $g(x) = \frac{x}{|x|}$.
3) $h(x) = \frac{2x-1}{x+1}$
- Do they allow extensions by continuity?

Solution: 1) - The definition domain of the function f defined by $f(x) = x\sin\left(\frac{1}{x}\right)$ est $D_f = \mathbb{R}^*$.

- The function f is continuous on $\mathbb{R}^*$(the composite and product of continuous functions). On the other hand, according to the proposition 10, we have

$$\lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = \lim_{x\to 0^-} x\sin\left(\frac{1}{x}\right) = 0.$$

Since $\lim\limits_{x\to 0} f(x)$ exists, then the function f admit an extension by continuity at $x_0 = 0$ and the extension function $\widetilde{f}$ is defined by

$$\widetilde{f}(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & if\ x \in I,\ (x \neq x_0), \\ 0, & if \quad x = x_0. \end{cases}$$

2) - The definition domain of the function g defined by $g(x) = \frac{x}{|x|}$ est $D_g = \mathbb{R}^*$.

- The function g is continuous on $\mathbb{R}^*$(the ratio of two functions continuous).
On the hand,

$$\lim_{x\to 0^+} g(x) = 1 \neq -1 = \lim_{x\to 0^-} g(x),$$

then g has not a limit at 0. Therefore g has not an extension by continuity at $x_0 = 0$.

3) - The definition domain of the function h defined by $h(x) = \frac{2x-1}{x+1}$ est $D_f = \mathbb{R} - \{-1\}$.

- The function h is continuous on $\mathbb{R} - \{-1\}$ (the ratio of two functions continuous).
On the hand,

$$\lim_{x\to(-1)^+} h(x) = -\infty \ and \lim_{x\to(-1)^-} h(x) = +\infty,$$

then h has not a limit at -1. Therefore h has not an extension by continuity at $x_0 = -1$.

Chapter 4

Comparaison near a point

Let f and g two functions defined on an interval I and $x_0 \in I$ (x_0 finite, infinite or extremity of I).

Definition 27. We say that f is dominate by g near x_0 if there exists $A > 0$ such that $|f(x)| \leq A|g(x)|$ for all x in neighborhood $J \subset I$ of x_0.
notation: $f = O(g)$.
If $g(x) \neq 0$, $\forall x \in J$, this means that $\frac{f}{g}$ is bounded over J.

Definition 28. Let f, g two functions defined on interval $]x_0, +\infty[$. We will suppose that $f = O(g)$ as $x \to +\infty \Longleftrightarrow \exists A > 0$, $\exists \delta > 0, \forall x\,[x > \delta \Rightarrow |f(x)| \leq A|g(x)|]$.

Definition 29. We say that f is negligible opposite g near x_0 if for all $\varepsilon > 0$ there is an neighborhood $J \subset I$ of x_0 such that we have $|f(x)| \leq \varepsilon|g(x)|$, $\forall x \in J$.
notation: $f = o(g)$ where $f \underset{x_0}{=} o(g)$.
If $g(x) \neq 0$,, $\forall x \in J$, this means that $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = 0$.

Definition 30. Let f, g two functions defined over interval $]x_0, +\infty[$. We will suppose that $f = O(g)$ as $x \to +\infty \Longleftrightarrow \exists \epsilon > 0$, $\exists \delta > 0, \forall x\,[x > \delta \Rightarrow |f(x)| \leq \epsilon|g(x)|]$.

Example 21. 1) $\lim\limits_{x \to +\infty} \frac{\ln(x)}{x} = 0 \Longleftrightarrow \ln(x) \underset{+\infty}{=} o(x)$.
2) $\lim\limits_{x \to +\infty} \frac{x}{e^x} = 0 \Longleftrightarrow x \underset{+\infty}{=} o(e^x)$.

4.1 Some properties of negligible functions

$o(f) + o(f) = o(f)$, $o(f) \cdot o(g) = o(fg)$, $o(o(f)) = o(f)$.
Furhter, if g is bounded, we have $g \cdot o(f) = o(f)$,

4.2 Fondamental properties

Approaches of $+\infty$, we have $(\ln(x))^\alpha = o(x)^\beta$ and $x^\alpha = o\left(e^{\beta x}\right)$ where $\alpha > 0$ and $\beta > 0$.
Approaches of 0, we have $|\ln x|^\alpha = o(x)^\beta$ where $\alpha > 0$ and $\beta < 0$.

Definition 31. We say that f and g are equivalente near x_0 if we have $f-g=o(g)$ for all $\varepsilon>0$ there exists an neighborhood J of x_0 such that we have $|f(x)-g(x)|\leq\varepsilon|g(x)|$, $\forall x\in J$.
notation: $f\underset{x_0}{\sim}g$.
If $g(x)\neq 0$, $\forall x\in J$, this means that $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=1$.

Remark 12. The relation $\sim$ is equivalence relation:
- Reflexive: $f\underset{x_0}{\sim}f$.
- Symmetric: If $f\underset{x_0}{\sim}g$ then $g\underset{x_0}{\sim}f$.
-Transitive: If $f\underset{x_0}{\sim}g$ and $g\underset{x_0}{\sim}h$, then $f\underset{x_0}{\sim}h$.

4.3 Properties of equivalence functions

1) If $f\underset{x_0}{\sim}g$ and $h\underset{x_0}{\sim}k$ then $f\times g\underset{x_0}{\sim}h\times k$, $|f|\underset{x_0}{\sim}|g|$ and $\frac{f}{g}\underset{x_0}{\sim}\frac{h}{k}$.
2) If $f\underset{x_0}{\sim}g$ and $\lim\limits_{x\to x_0}g(x)=l$ then $f\underset{x_0}{\sim}l$.
3) If $f\underset{x_0}{\sim}g$, $f>0$ and $g>0$ then $f^\alpha\underset{x_0}{\sim}g^\alpha$, $\forall\alpha\in\mathbb{R}$.

Remark 13. 1) In general $f+h\underset{x_0}{\sim}g+k$ and $f-h\underset{x_0}{\sim}g-k$ are not satisfied, since $f\underset{x_0}{\sim}g$.
2) $|f|\underset{x_0}{\sim}|g|$ not implies $f\underset{x_0}{\sim}g$.

Example 22. 1) According to the remark 6, we have the following examples :

$$(1)\begin{cases}-x\underset{0}{\sim}-x\\ \sin(x)\underset{0}{\sim}x\end{cases}\quad\Rightarrow\sin(x)-x\underset{0}{\sim}0\ false.$$

$$(2)\begin{cases}x^2\underset{0}{\sim}x^2+\frac{1}{x}\\ -x^2+x\underset{0}{\sim}-x^2\end{cases}\quad\Rightarrow x^2-x^2+x\underset{0}{\sim}\frac{1}{x}\ false.$$

2) $\sqrt{x^2-16}\underset{4^+}{\sim}\sqrt{8}\sqrt{x-4}$, because, we have $\sqrt{x^2-16}=\sqrt{(x-4)(x+4)}$, and according to the properties (1) and (3), we have

$$\begin{cases}x-4\underset{4}{\sim}x-4\\ x+4\underset{4}{\sim}8\end{cases}\quad\Rightarrow(x-4)(x+4)\underset{4}{\sim}8(x-4)$$

$$\Rightarrow\sqrt{(x-4)(x+4)}\underset{4^+}{\sim}\sqrt{8}\sqrt{x-4}.$$

Proposition 11. *If $f\underset{x_0}{\sim}g$, $h\underset{x_0}{\sim}k$, h and k are strictly positives (or strictly negatives), then $f+h\underset{x_0}{\sim}g+k$.*

Example 23. See exercice $15,(b)$.

4.4 Classic equivalences:

Approaches of 0: $\ln(1+x) \underset{0}{\sim} x$, $e^x - 1 \underset{0}{\sim} x$, $1 - \cos(x) \underset{0}{\sim} \frac{x^2}{2}$, $\sin(x) \underset{0}{\sim} x$, $\tan(x) \underset{0}{\sim} x$ and $(1+x)^\alpha - 1 \underset{0}{\sim} \alpha x$.

Proposition 12. *If $f \underset{x_0}{=} o(g)$ then $f + g \underset{x_0}{\sim} g$.*

Proof. See exercice 17. □

Example 24. 1) $x + e^x \underset{+\infty}{\sim} e^x$ because, since $x \underset{+\infty}{=} o(e^x)$, then $x + e^x \underset{+\infty}{\sim} e^x$.

2) $\frac{1}{x} + \ln(x) \underset{0}{\sim} \frac{1}{x}$ because, since $ln(x) \underset{0}{=} o\left(\frac{1}{x}\right)$, then $\frac{1}{x} + \ln(x) \underset{0}{\sim} \frac{1}{x}$.

3) $\ln(1+x) \underset{+\infty}{\sim} \ln(x)$ because, we have

$$\ln(1+x) = \ln\left[x\left(\frac{1}{x}+1\right)\right] = \ln(x) + \ln\left(1+\frac{1}{x}\right).$$

So, and since $\ln\left(1+\frac{1}{x}\right) \underset{+\infty}{=} o(\ln(x))$, then $\ln(1+x) \underset{+\infty}{\sim} \ln(x)$.

Proposition 13. *Si $\lim\limits_{x\to x_0} f(x) = l$ et $\lim\limits_{x\to x_0} g(x) = l$, alors $f \underset{x_0}{\sim} g$ fausse.*
Contre exemple: On a

$$\begin{cases} f(x) = x, & \lim\limits_{x\to 0} f(x) = 0, \\ g(x) = 2x, & \lim\limits_{x\to 0} g(x) = 0, \end{cases}$$

mais

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{1}{2}.$$

That is f et g are not equivalentes near $x_0 = 0$.

Proposition 14. *If $\lim\limits_{x\to x_0} f(x) = +\infty$ and $\lim\limits_{x\to x_0} g(x) = +\infty$, then $f \underset{x_0}{\sim} g$ false.*
Indeed:

$$\begin{cases} f(x) = x^2, & \lim\limits_{x\to +\infty} f(x) = +\infty, \\ g(x) = x^3, & \lim\limits_{x\to +\infty} g(x) = +\infty. \end{cases}$$

But

$$\lim_{x\to +\infty} \frac{f(x)}{g(x)} = \lim_{x\to +\infty} \frac{1}{x} = 0.$$

That is f and g are not equivalentes near $x_0 = +\infty$.

Theorem 12. *Let f and g two functions defined near 0 and h the function defined near x_0, then*

$$\begin{cases} f(t) \underset{0}{\sim} g(t) \\ \lim\limits_{x\to x_0} h(x) = 0 \end{cases} \Rightarrow f(h(x)) \underset{x_0}{\sim} g(h(x)).$$

Example 25. 1) We have, $\ln(x) \underset{1}{\sim} -1+x$, because $\ln(x) = \ln[1+(x-1)]$, if we set $h(x) = x-1$, we have $h(x) \to 0 \ as \ x \to 1$ and we have

$$\ln(x) = \ln[1+(x-1)] \underset{1}{\sim} -1+x.$$

2) We have $\ln(1+x^2+2x) \underset{0}{\sim} x^2+x$, because if we set $h(x) = x^2+2x$, we have $h(x) \to 0 \ as \ x \to 0$ and we have

$$\ln\left(1+x^2+2x\right) \underset{0}{\sim} x^2+2x.$$

Exercise 14. 1- Prove the following equalities:

a) $2x^3 - 3x^2 \underset{0}{=} o(x)$,
b) $(1+x)^n - (1+nx) \underset{0}{=} o(x) \ si \ n \in \mathbb{N}^*$.
2 - Are the following propositions true or false:
a) $x \underset{0^+}{=} o(\sqrt{x})$,
b) $\sin(x) - x \underset{0}{=} o(x)$,
c) $\ln(1+x) - x \underset{0}{=} o(1)$,
d) $x^2 \underset{+\infty}{=} o(x^2+1)$,
e) $o(x^2) + o(x) \underset{0}{=} o(x)$.

Solution: a) Since $\lim\limits_{x\to 0} \frac{2x^3-3x^2}{x} = \lim\limits_{x\to 0} [2x^2-3x] = 0$, then $2x^3 - 3x^2 \underset{0}{=} o(x)$.
b) We will prove $(1+x)^n - (1+nx) \underset{0}{=} o(x) \ if \ n \in \mathbb{N}^*$. We have

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots+\frac{n(n-1)\cdots(n-p+1)}{p!}x^p+\cdots+x^n$$

$$= 1+nx+x\left[\frac{n(n-1)}{2}x+\frac{n(n-1)(n-2)}{3!}x^2+\cdots+\frac{n(n-1)\cdots(n-p+1)}{p!}x^{p-1}+\cdots+x^{n-1}\right],$$

So,

$$(1+x)^n-1+nx = x\left[\frac{n(n-1)}{2}x+\frac{n(n-1)(n-2)}{3!}x^2+\cdots+\frac{n(n-1)\cdots(n-p+1)}{p!}x^{p-1}+\cdots+x^{n-1}\right]$$

Therefore, since

$$\lim_{n\to 0}\left[\frac{n(n-1)}{2}x+\frac{n(n-1)(n-2)}{3!}x^2+\cdots+\frac{n(n-1)\cdots(n-p+1)}{p!}x^{p-1}+\cdots+x^{n-1}\right] = 0,$$

then $(1+x)^n - (1+nx) \underset{0}{=} o(x) \ si \ n \in \mathbb{N}^*$.
2) a) $x \underset{0^+}{=} o(\sqrt{x})$ true . b) $\sin(x) - x \underset{0}{=} o(x)$ true. c) $\ln(1+x) - x \underset{0}{=} o(1)$ true. d) $x^2 \underset{+\infty}{=} o(x^2+1)$ true. e) $o(x^2) + o(x) \underset{0}{=} o(x)$ true.

Exercise 15. 1- Show that

a) $\frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_nx^n} \underset{+\infty}{\sim} \frac{a_n}{b_n}$, where $a_i, \ b_i \in \mathbb{R}, \ i \in \{0,1,\ldots,n\}, \ n \in \mathbb{N}, \ b_n \neq 0$.
b) $\frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_nx^n} \underset{0}{\sim} \frac{a_0}{b_0}$, where $a_i, \ b_i \in \mathbb{R}, \ i \in \{0,1,\ldots,n\}, \ n \in \mathbb{N}, \ b_0 \neq 0$.

Solution: a) Let $b_n \neq 0$. We have

$$\lim_{x\to+\infty} \frac{\frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_nx^n}}{\frac{a_n}{b_n}} = \lim_{x\to+\infty} \frac{\frac{a_nx^n}{b_nx^n}}{\frac{a_n}{b_n}}$$

$$= \lim_{x\to+\infty} \frac{\frac{a_n}{b_n}}{\frac{a_n}{b_n}} = 1.$$

So,

$$\frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_nx^n} \underset{+\infty}{\sim} \frac{a_n}{b_n}.$$

b) Let $b_0 \neq 0$. We have

$$\lim_{x\to 0} \frac{\frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_nx^n}}{\frac{a_0}{b_0}} = \lim_{x\to 0} \frac{\frac{a_0}{b_0}}{\frac{a_0}{b_0}} = 1.$$

So,

$$\frac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\cdots+b_nx^n} \underset{0}{\sim} \frac{a_0}{b_0}.$$

Exercise 16. - Are the following propositions true or false:

a) $x^3 + \frac{x}{1-x} - \frac{x}{1+x} \underset{0}{\sim} x^3$.

b) $x + \sqrt{1+x^2} \underset{+\infty}{\sim} 2x$.

c) $x^2 \left(\tan\left(\frac{\pi}{x}\right)\right)^2 \underset{+\infty}{\sim} \pi^2$.

solution: a) $x^3 + \frac{x}{1-x} - \frac{x}{1+x} \underset{0}{\sim} x^3$ false, because

$$\lim_{x\to 0} \frac{x^3 + \frac{x}{1-x} - \frac{x}{1+x}}{x^3} = \lim_{x\to 0} \frac{x^3 + \frac{2x^2}{x^2-1}}{x^3}$$

$$= \lim_{x\to 0} \frac{x^5 - x^3 + 2x^2}{x^5 - x^3}$$

$$= \lim_{x\to 0} \frac{x^2 - x + \frac{2}{x}}{x^2-1} = \pm\infty \neq 1.$$

b) $x + \sqrt{1+x^2} \underset{+\infty}{\sim} 2x$, true, because

$$\begin{cases} \sqrt{x^2+1} \underset{+\infty}{\sim} \sqrt{x^2} = |x| = x, \\ x \underset{+\infty}{\sim} x, \end{cases}$$

and since x is strictly positive near $+\infty$, then $x + \sqrt{1+x^2} \underset{+\infty}{\sim} 2x$.

c) $x^2 \left(\tan\left(\frac{\pi}{x}\right)\right)^2 \underset{+\infty}{\sim} \pi^2$ true, because according to the product theorem, we have

$$\begin{cases} \left(\tan\left(\frac{\pi}{x}\right)\right)^2 \underset{+\infty}{\sim} \left(\frac{\pi}{x}\right)^2, \\ x^2 \underset{+\infty}{\sim} x^2. \end{cases} \Rightarrow x^2 \left(\tan\left(\frac{\pi}{x}\right)\right)^2 \underset{+\infty}{\sim} \pi^2.$$

Exercise 17. Let f, g two functions defined and positives in neighbohood of x_0. We suppose that $f \underset{x_0}{\sim} g$.

a) A-t-on $\ln f \underset{x_0}{\sim} \ln g,\ f,\ g > 0$.
b) Show that if $\lim\limits_{x\to x_0} g(x) = l$ where $l \in [0,1[\cup]1,+\infty]$ then $\ln f \underset{x_0}{\sim} \ln g$.
c) A-t-on $e^f \underset{x_0}{\sim} e^g$.
d) Show that $e^f \underset{x_0}{\sim} e^g$ if and only if $\lim\limits_{x\to x_0} (f(x) - g(x)) = 0$.
e) Show that $\sqrt[n]{f} \underset{x_0}{\sim} \sqrt[n]{g},\ n \in \mathbb{N}$.
f) Show that, if $f \underset{x_0}{=} o(g)$ then $f + g \underset{x_0}{\sim} g$.

Solution: a) In general $\ln f \underset{x_0}{\sim} \ln g$ false. We can consider the following example :

$f(x) = 1 + \frac{x^2}{2} + \sin^2(x)\ ,\ g(x) = 1 + x^2,\ et\ x_0 = 0.$

On the other hand, $\lim\limits_{x\to 0}\frac{f(x)}{g(x)} = \lim\limits_{x\to 0}\frac{1+\frac{x^2}{2}+\sin^2(x)}{1+x^2} = 1$ i.e, $f \underset{0}{\sim} g$ and $\ln\left(1 + \frac{x^2}{2} + \sin^2(x)\right) \underset{0}{\sim} \frac{x^2}{2} + \sin^2(x)\ ,\ \ln g\left(1 + x^2\right) \underset{0}{\sim} x^2.$

Therefor, $\lim\limits_{x\to 0}\frac{\ln f(x)}{\ln g(x)} = \lim\limits_{x\to 0}\frac{\frac{x^2}{2}+\sin^2(x)}{x^2} = \lim\limits_{x\to 0}\left(\frac{1}{2} + \left(\frac{\sin(x)}{x}\right)^2\right) = \frac{3}{2}$, that is $\ln f$ and $\ln g$ is not quivalent at point 0.

b) We will prove that, if $\lim\limits_{x\to x_0} g(x) = l$ où $l \in [0,1[\cup]1,+\infty]$ then $\ln f \underset{x_0}{\sim} \ln g$ et $f,\ g > 0$.

Let $f \underset{x_0}{\sim} g$. We have

$$\begin{aligned}
\lim_{x\to x_0}\frac{\ln f(x)}{\ln g(x)} &= \lim_{x\to x_0}\frac{\ln\frac{f(x)\times g(x)}{g(x)}}{\ln g(x)} \\
&= \lim_{x\to x_0}\frac{\ln\frac{f(x)\times g(x)}{g(x)}}{\ln g(x)} \\
&= \lim_{x\to x_0}\frac{\ln\left(\frac{f(x)}{g(x)}\times g(x)\right)}{\ln g(x)} \\
&= \lim_{x\to x_0}\frac{\ln\frac{f(x)}{g(x)} + \ln g(x)}{\ln g(x)} \\
&= \lim_{x\to x_0}\left[\frac{\ln\frac{f(x)}{g(x)}}{\ln g(x)} + 1\right].
\end{aligned}$$

Since $\ln f \underset{x_0}{\sim} \ln g \Longleftrightarrow \lim\limits_{x\to x_0}\frac{\ln f(x)}{\ln g(x)} = 1$, it sufficient

$$\lim_{x\to x_0}\left[\frac{\ln\frac{f(x)}{g(x)}}{\ln g(x)} + 1\right] = 1.$$

So, we have the following possibilities:

$$\begin{cases} \lim\limits_{x\to x_0} \ln\left[\frac{f(x)}{g(x)}\right] = 0, \ \ et \ \lim\limits_{x\to x_0} \ln g\left(x\right) = l \neq 0, \\ \qquad\qquad ou \\ \lim\limits_{x\to x_0} \ln g\left(x\right) = +\infty, \end{cases}$$

So,

$$\begin{cases} \lim\limits_{x\to x_{x_0}} f\left(x\right) = 0, \ \ et \lim\limits_{x\to x_0} \ln g\left(x\right) = l \neq 0, \\ \qquad\qquad\qquad ou \\ \qquad \lim\limits_{x\to x_0} g\left(x\right) = +\infty. \end{cases}$$

On the other hand,

$$\begin{aligned} \lim_{x\to x_0} \left[\frac{\ln \frac{f(x)}{g(x)}}{\ln g\left(x\right)} + 1\right] \neq 1 &\iff \lim_{x\to x_0} \ln g\left(x\right) = 0 \\ &\iff \lim_{x\to x_0} g\left(x\right) = 1. \end{aligned}$$

c) $e^f \underset{x_0}{\sim} e^g$ false. We can consider the following example :

$f\left(x\right) = \frac{1}{x^4}, \ g\left(x\right) = \frac{1}{x^4} + \frac{1}{x^2}, \ and \ x_0 = 0.$

On the other hand, $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = \lim\limits_{x\to 0} \frac{\frac{1}{x^4}}{\frac{1}{x^4}+\frac{1}{x^2}} = \lim\limits_{x\to 0} \frac{1}{1+x^2} = 1$ i.e., $f \underset{0}{\sim} g$ and $\lim\limits_{x\to 0} \frac{e^{f(x)}}{e^{g(x)}} = \lim\limits_{x\to 0} e^{-\frac{1}{x^2}} = 0$, that is e^f et e^g is not equivalent at point 0.

d) We will show that, $e^f \underset{x_0}{\sim} e^g$ if and only if $\lim\limits_{x\to x_0} \left(f\left(x\right) - g\left(x\right)\right) = 0$. We have

$$\begin{aligned} e^f \underset{x_0}{\sim} e^g &\iff \lim_{x\to x_0} \frac{e^{f(x)}}{e^{g(x)}} = 1 \\ &\iff \lim_{x\to x_0} e^{f(x)} \times e^{-g(x)} = 1 \\ &\iff \lim_{x\to x_0} e^{f(x)-g(x)} = 1 \\ &\iff \lim_{x\to x_0} e^{f(x)-g(x)} = e^0 \\ &\iff e^{\lim\limits_{x\to x_0} [f(x)-g(x)]} = e^0 \\ &\iff \lim_{x\to x_0} \left[f\left(x\right) - g\left(x\right)\right] = 0. \end{aligned}$$

f) We will prove that, if $f \underset{x_0}{=} o\left(g\right)$ then $f + g \underset{x_0}{\sim} g$.

Let $f \underset{x_0}{=} o\left(g\right)$. We have

$$\lim_{x\to x_0} \frac{f\left(x\right) + g\left(x\right)}{g\left(x\right)} = \lim_{x\to x_0} \left[\frac{f\left(x\right)}{g\left(x\right)} + 1\right],$$

so, since $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = 0$, then

$$\lim_{x\to x_0} \frac{f\left(x\right) + g\left(x\right)}{g\left(x\right)} = \lim_{x\to x_0} \left[\frac{f\left(x\right)}{g\left(x\right)} + 1\right] = 1.$$

Exercise 18. Determine the following limits (if there exists):

1) $\lim\limits_{x\to 0}\frac{e^{x}-e^{x^{2}}}{x^{2}-x}$.
2) $\lim\limits_{x\to +\infty}\frac{x+1+\ln(x)}{\ln(x)}$.
3) $\lim\limits_{x\to 0}\frac{x+1+\ln(x)}{\ln(x)}$.

Solution: For calculate the following limits we use the properties of equivalente functions.

1) Since $e^{x^2}-e^{x}=e^{x}\left(e^{x}-1\right)$, then we have

$$\begin{cases} e^{x}\underset{0}{\sim}1 \\ e^{x}-1\underset{0}{\sim}x \end{cases} \Rightarrow e^{x}\left(e^{x}-1\right)\underset{0}{\sim}x\ .$$

So,

$$\begin{cases} e^{x^2}-e^{x}\underset{0}{\sim}x \\ x^{2}-x\underset{0}{\sim}-x \end{cases} \Rightarrow \frac{e^{x^2}-e^{x}}{x^{2}-x}\underset{0}{\sim}-1\ .$$

Therefore

$$\lim_{x\to 0}\frac{e^{x}-e^{x^{2}}}{x^{2}-x}=-1.$$

2) Since $\ln\left(x\right)\underset{+\infty}{=}o\left(x\right)$, then

$$\frac{x+1+\ln\left(x\right)}{\ln\left(x\right)}\underset{+\infty}{\sim}\frac{1+x}{\ln\left(x\right)}\underset{+\infty}{\sim}\frac{x}{\ln\left(x\right)}.$$

So,

$$\lim_{x\to+\infty}\frac{x+1+\ln\left(x\right)}{\ln\left(x\right)}=\lim_{x\to+\infty}\frac{x}{\ln\left(x\right)}=+\infty.$$

3) Since $x\underset{0}{=}o\left(\ln\left(x\right)\right)$, then

$$\frac{x+1+\ln\left(x\right)}{\ln\left(x\right)}\underset{0}{\sim}\frac{1+\ln\left(x\right)}{\ln\left(x\right)}\underset{0}{\sim}\frac{\ln\left(x\right)}{\ln\left(x\right)}=1.$$

So,

$$\lim_{x\to 0}\frac{x+1+\ln\left(x\right)}{\ln\left(x\right)}=1.$$

Chapter 5

Differentiable functions

5.1 Generality

5.1.1 Differentiability of functions at point

Definition 32. Let I an open interval of $\mathbb{R}$, $x_0 \in I$ and a function $f : I \to \mathbb{R}$. We say that f is differentiable at x_0 if there exists a real number l (l finite) such that

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0)}{x - x_0} = l,$$

l is called the derivative of f at point x_0, is denoted by $f'(x_0)$ (derivative number).

Example 26. We will study the differentiability of the function $f : x \to x^3 - 4x$ at $x_0 = 1$: We have $D_f = \mathbb{R}$ et $f(1) = -3$,

$$\begin{aligned} \lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{\substack{x \to 1 \\ x \neq 1}} \frac{x^3 - 4x + 3}{x - 1} \\ &= \lim_{\substack{x \to 1 \\ x \neq 1}} \frac{(x-1)(x^2 + x - 3)}{x - 1} = -1, \end{aligned}$$

So, f is differentiable at $x_0 = 1$ et on a $f'(x_0) = f^{'}(1) = -1$.

Exercise 19. Study the differentiability at x_0 of the following functions:
1) $g(x) = \frac{-x}{2x-1}$, $x_0 = 0$,
2) $h(x) = \frac{x^4-4x}{x-2}$, $x_0 = 1$,
3) $k(x) = \sqrt{x^2 + 9x}$, $x_0 = 2$
4) $l(x) = x^2 + 5x$, $x_0 = \alpha$.

5.1.2 Derivative function

Proposition 15. *The function f is said to be differentiable over open interval I if is differentiable at evry point $x_0 \in I$.*

The derivative function of f is defined over I by $f' : x \to f'(x)$.

5.1.3 Left and right derivative (One-sided derivatives)

Let x_0 such that $[x_0, x_0+\varepsilon[\subset I$ (respectively $]x_0-\varepsilon, x_0] \subset I$) for $\epsilon > 0$.

The function f is deffirentiable at right if

$$\lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = l_1 \in \mathbb{R},$$

and $l_1 = f'_d(x_0)$ is called right-hand derivative of f at x_0.

The function f is differentiable at left if

$$\lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = l_2 \in \mathbb{R},$$

and $l_2 = f'_g(x_0)$ is called left-hand derivative of f at x_0.

Proposition 16. *The function f is differentiable at x_0 if and only if f admit (has) at x_0 an right-hand and left-hand derivative are equal and finites (exist and are equal).*

Remark 14. If $f'_d(x_0)$ and $f'_g(x_0)$ exist and $f'_d(x_0) \neq f'_g(x_0)$ then f is not differentiable at x_0.

Example 27. The function $f : x \longmapsto |x|$ is not differentiable at the origin $x_0 = 0$. To see this fact, we have

$$\lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to 0^+} \frac{x}{x} = 1,$$

and on other hand

$$\lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to 0^-} \frac{-x}{x} = -1,$$

thus, since $f'_d(0) \neq f'_g(0)$, then the function f is not differentiable at the origin $x_0 = 0$.
The function $g : x \longmapsto \sqrt{x}$ is not differentiable at the origin $x_0 = 0$. To see this fact, we have

$$\lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} = +\infty,$$

therefore, since the limit of ratio is infinitely, then the function f is not differentiable at the origin $x_0 = 0$.

A famous example of Weierstrass shows that there exists a continuous function that is not differentiable at any point. The construction of this function is beyond the scope of this chapter. On the other hand, a differentiable function is always continuous.

Proposition 17. *((Differentiability implies continuity) If $f : I \to \mathbb{R}$ is differentiable at point $x_0 \in I$. Then, f is continuous at x_0.*

Exercise 20. Let the function f defined by

$$f(x) = \begin{cases} \frac{|x|}{x} + x - 1, & if \;\; x \neq 0, \\ 0, & if \; x = 0. \end{cases}$$

1) Study the differentiable of the function f at $x_0 = 0$.
1) Determine the set of the differentiable of the function f.

Exercise 21. Let the function f defined by

$$f(x) = \begin{cases} \frac{x-1}{x+1}, & if\ x > -1, \\ x^3 - 4, & if\ x \leq -1. \end{cases}$$

1) Determine the domain of definition of the function f.
2) Study the differentiable of the function at $x_0 = -1$.
3) Find the set of the differentiable of the function f.

Solution: 1) $D_f = \mathbb{R}$.

2) We will study the differentiability of the following functions at $x_0 = -1$. We have $f(-1) = -5$ and

$$\lim_{x \to (-1)^+} \frac{f(x) - f(-1)}{x+1} = -\infty,$$

So, f is not differentible at right at $x_0 = -1$. And

$$\lim_{x \to (-1)^-} \frac{f(x) - f(-1)}{x+1} = -1,$$

So, f is differentiable at left at $x_0 = -1$.

3) Since f $x_0 = -1$ is differentiable at left at and not differentible at right at $x_0 = -1$. Then, the set of differentiability, of the function f is $]-\infty, -1] \cup]-1, +\infty[$.

Exercise 22. Are the following functions differentiable:

$$f(x) = \begin{cases} x^2 e^{-x^2}, & if\ |x| \leq 1, \\ 1, & if\ |x| > 1. \end{cases}$$

$$g(x) = \left|\pi^2 - x^2\right| \sin^2(x).$$

5.1.4 Geometric interpretation

Le plan étant muni par un repère orthonormal .
La fonction f est dérivable en x_0 signifie que le graphe de f admet au point d'abscisse x_0 une tangante de pente $f'(x_0)$.
1) If $\lim\limits_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x)-f(x_0)}{x-x_0} = \pm\infty$. Then, the graphe admit a vertical tangente at point $(x_0, f'(x_0))$.
2) If f continuous at x_0 and $f'_d(x_0) \neq f'_g(x_0)$. Then the graph has angle point at x_0.

5.2 Basic rules for differentiation

5.2.1 Derivative of Composite function (Chain rule for differentiation)

Proposition 18. *Let $f : I \to J$, $g : J \to \mathbb{R}$ and $x_0 \in I$.*

If f is differentiable at x_0 and if g is differentiable at $f(x_0)$. Then $g \circ f$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0).$$

Example 28. 1) Let $f(x) = \sin^5(x)$. To find the derivative of f we can simply apply the chain rule, we have

$$f'(x) = 5\cos(x)\sin^4(x).$$

Notice that it is quite easy to extend this to any power. Set $f(x) = \sin^n(x)$, $n \in \mathbb{N}^*$. Then follow the same steps and we arrive at

$$f'(x) = n\cos(x)\sin^{n-1}(x).$$

2) Let $a, b \in \mathbb{R}$ and let $f(ax+b)$ be a differentiable function. Then

$$f'(x) = af'(ax+b).$$

The above is a very useful result that follows from the chain rule, so lets make it a the following two examples :

$$f(x) = \cos(ax+b) \ \ \textit{and} \quad h(x) = \sin(ax+b).$$

Then

$$f'(x) = -a\sin(ax+b) \ \ \textit{and} \quad h'(x) = a\cos(ax+b).$$

3) Let $f(x) = \sqrt[3]{\sin(x^2)} = (\sin(x^2))^{\frac{1}{3}}$. Then

$$f'(x) = \frac{2x\cos(x^2)}{3(\sin(x^2))^{\frac{2}{3}}}.$$

5.2.2 Derivative of inverse function

Proposition 19. *Let $f : I \to J$ is bijection function, $x_0 \in I$ and $y_0 = f(x_0) \in J$. Suppose that f is differentiable at x_0, $f'(x_0) \neq 0$ and que f^{-1} is continuous at y_0. Then*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{(f' \circ f^{-1})(y_0)}.$$

If f is continuously differentiable and f' is never zero, then f^{-1} is continuously differentiable.

5.3 Fondamental theorems

5.3.1 Rolle's theorem

Theorem 13. *Let a and b be real numbers with $a < b$. If f is continuous function in closed interval $[a, b]$ and f is differentiable on the open interval $]a, b[$and if $f(a) = f(b)$. Then, there is $c \in]a, b[$ such that $f'(c) = 0$.*

5.3.2 Mean-value theorem

Theorem 14. *Let a and b be real numbers with $a < b$. If f is continuous function in closed interval $[a, b]$ and f is differentiable on the open interval $]a, b[$. Then, there is $c \in]a, b[$ such that*

$$f^{'}(b) - f(a) = (b-a) f^{'}(c),$$

which we can also express as

$$f^{'}(c) = \frac{f^{'}(b) - f(a)}{b-a}.$$

Another formulation of mean-value therem:

Theorem 15. *Let $I \subset \mathbb{R}$, $a \in I$ and f continuous and differentiable on I except possibly at a. Then, for all h verifying $a + h \in I$, there is $\theta \in]0, 1[$ such that*

$$f(a+h) - f(a) = hf^{'}(a + \theta h).$$

5.3.3 Generalised mean-value theorem

Theorem 16. *Let f and g both be defined and continuous functions on $[a, b]$ and both be differentiable on $]a, b[$. Furthermore, suppose that $g'(x) \neq 0$, $\forall x \in]a, b[$. Then there is $c \in]a, b[$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f^{'}(c)}{g'(c)}$.*

5.3.4 Hôpital's rule

Theorem 17. *Let $I \subset \mathbb{R}$, $x_0 \in \mathbb{R} \cup \{-\infty, +\infty\}$, f, g be two functions definied and continuous on I except possiply at x_0 such that*

(i) $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$ or $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \pm\infty$,

(ii) f and g are differentiable on $I \setminus \{x_0\}$and $g'(x) \neq 0$, $\forall x \in I \setminus \{x_0\}$.

if $\lim\limits_{x\to x_0} \frac{f^{'}(x)}{g'(x)} = l$ (where $l \in \mathbb{R} \cup \{-\infty, +\infty\}$), then $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = l$.

Remark 15. We can apply L'Hôpital's rule a second time ect.

Remark 16. Reciprocly (Conversely) of Theorem 17 is false.

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & si\ x \neq 0, \\ 0, & si\ x = 0, \end{cases} \quad \text{et } g(x) = \sin(x) \text{ et } x_0 = 0.$$

We have

$\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = \lim\limits_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sin(x)} = 0$,

but $\lim\limits_{x\to 0} \frac{f^{'}(x)}{g'(x)} = \lim\limits_{x\to 0} \frac{1}{\cos(x)} \left[2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right]$ not exists.

Proposition 20. *(Differential). Let f be a function differentiable at $x \in \mathbb{R}$ such that $y = f(x)$. Let dx be a (small) given number. The differential dy is defined as*

$$dy = f'(x)\, dx.$$

Exercise 23. Epxplain how you can use mean value theorem to the follwing function :

Verify that which function satisfies the hypothesis of the mean value theorem on the interval given. Then find all numbers c that satisfy the conclusion of the mean value theorem. Leave your final answer(s) exact.

1) $f(x) = 2(x^2 - 5)$ on $[3, 5]$.

2) $g(x) = |x - 1|$ on $[0, 2]$.

3) $h(x) = \begin{cases} x, & if\ 0 \leq x \leq 1, \\ 2 - x, & if\ 1 < x \leq 2, \end{cases}$ sur $[0, 2]$.

4) $k(x) = \begin{cases} \frac{x^2-3}{2}, & if\ x \leq 1, \\ \frac{-1}{x}, & if\ x > 1. \end{cases}$ sur $[0, 2]$.

Solution: We will using the mean value theorem to the following functions (if its possible):

1) $f(x) = 2(x^2 - 5)$ sur $[3, 5]$. On a $f(3) = 8$, $f(5) = 40$ et $f'(x) = 4x$.

Since f is continuous on $[3, 5]$ and differentiable over $[3, 5]$, in particular on $]3, 5[$, then according to the mean value theorem $\exists c \in]3, 5[$such that

$$\begin{aligned} f(b) - f(a) = (b - a) f^{'}(c) \Rightarrow f(5) - f(3) &= (5 - 3)(4c) \\ &\Rightarrow 40 - 8 = 2(4c) \\ &\Rightarrow c = 4, \end{aligned}$$

we have $3 < c < 5$.

2) Since the function g is not differentiable on $]0, 2[$ exactly at $x = 1$, then we can not applied the mean value theorem to the function g.

3) Since the function h is not differentiable on $]0, 2[$ exactly at $x = 1$, then we can not applied the mean value theorem to the function h.

4) The mean value theorem is applied and we get $c = 1$.

Exercise 24. Using the the mean value theorem to the functions $f(x) = \cos(x)$ over the l'interval $\left[\frac{59\pi}{180}, \frac{\pi}{3}\right]$ for calculate $\cos\left(\frac{59\pi}{180}\right)$.

- The same question, for $\sin\left(\frac{29\pi}{180}\right)$.

Solution: Since f is continuous and differentiable on $\mathbb{R}$, in particular$\left[\frac{59\pi}{180}, \frac{\pi}{3}\right]$, then according to the mean value theorem, we have

$$\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{59\pi}{180}\right) = -\frac{\pi}{180} \sin(\theta), \quad \frac{59\pi}{180} < \theta < \frac{\pi}{3},$$

that is

$$\cos\left(\frac{59\pi}{180}\right) = \frac{1}{2} + \frac{\pi}{180} \sin(\theta), \quad \frac{59\pi}{180} < \theta < \frac{\pi}{3},$$

since $\frac{\pi}{4} < \theta < \frac{\pi}{3}$, then $\frac{\sqrt{2}}{2} < \sin(\theta) < \frac{\sqrt{3}}{2}$, we have

$$\frac{1}{2} + \frac{\pi\sqrt{2}}{360} < \cos\left(\frac{59\pi}{180}\right) < \frac{1}{2} + \frac{\pi\sqrt{3}}{360},$$

or,

$$0,512 < \cos\left(\frac{59\pi}{180}\right) < 0,516,$$

Therefore, à $\frac{1}{1000}$ près par défaut

$$\cos\left(\frac{59\pi}{180}\right) \cong 0,515,$$

Exercise 25. Use the mean value theorem, prove that

$$1 - x < e^x < \frac{1}{1-x}, \quad \forall x \in]0,1[$$

$$x - \frac{x^3}{6} < \sin(x) < x, \quad \forall x > 0.$$

Solution: We will use the the mean value theorem to the following functions :

1) To show the inequality $1 - x < e^x, \quad \forall x \in]0,1[$, we consider the function f defined by $f(t) = e^t + t - 1$.

Since f is differentiable on $\mathbb{R}$, we can apply the the mean value theorem on any closed bounded interval on $\mathbb{R}$.

Let $x \in]0,1[$, we will applied the the mean value theorem to f over $[0,x]$. According to the the mean value theorem $\exists c \in]0,x[$ such that

$$f(x) - f(0) = (x-0)f'(c) \Rightarrow e^x + x - 1 = x(e^c + 1)$$

$$\Rightarrow e^x - 1 = xe^c.$$

Since $\exists c \in]0,x[$, then $0 < c < x$, So, since the function $t \longmapsto e^t$ is strictly decreasing on $\mathbb{R}$, in particular on $]0,x[\subset \mathbb{R}$, then we have

$$0 < c < x \Rightarrow 1 < e^c < e^x$$

$$\Rightarrow 1 < e^c < e^x$$

$$\Rightarrow x < xe^c < xe^x, \quad car\ x > 0$$

$$\Rightarrow -x < x < xe^c < xe^x, \quad car\ x > 0.$$

Thus, from $e^x - 1 = xe^c$ and $-x < x < xe^c < xe^x$, we have

$$-x < e^x - 1 \Rightarrow 1 - x < e^x.$$

Similarly, To prove the inequality $e^x < \frac{1}{1-x}, \quad \forall x \in]0,1[$, we consider the function f defined by $g(t) = e^t(1-t) - 1$.

Since f is differentiable on $\mathbb{R}$, then we can apply the mean value theorem on any bounded closed interval on $\mathbb{R}$.

Let $x \in]0,1[$. We will apply the theorem on f on $[0,x]$. The mean value theorem implies that $\exists c \in]0,x[$ such that

$$g(x) - g(0) = (x-0)g'(c) \Rightarrow (1-x)e^x - 1 = -cxe^c.$$

Since $0 < c < x$, and $e^t > 0,\ \forall t \in \mathbb{R}$, particulary on $]0, x[\subset \mathbb{R}$, then we have

$$(1-x)\,e^x - 1 = -cxe^c < 0,$$

so,

$$\begin{aligned}(1-x)\,e^x - 1 < 0 &\Rightarrow (1-x)\,e^x < 1\\ &\Rightarrow (1-x)\,e^x < 1\\ &\Rightarrow e^x < \frac{1}{1-x}.\end{aligned}$$

Therefore,

$$1 - x < e^x < \frac{1}{1-x}, \quad \forall x \in \left]0, 1\right[.$$

Exercise 26. Using the l'Hôspital's rule to calculate thevfollowing limits :

1) $\lim\limits_{x\to 0} \frac{\ln\left(\frac{x^2+1}{x^2}\right)}{e^x\left(x+\frac{2}{x}\right)}$

2) $\lim\limits_{x\to 0} \frac{x^x-1}{1-x+\ln(x)}$

3) $\lim\limits_{x\to 0} \frac{\left(1+\frac{1}{x}\right)^x-e}{x}$

4) $\lim\limits_{x\to 0} \frac{1-\cos\left(x^2\right)}{x^4}$

Solution: Sometimes it pays to combine L'Hospital's rule with other manipulations.

1) We set : $h(x) = \frac{h_1(x)}{h_2(x)} = \frac{\ln\left(\frac{x^2+1}{x^2}\right)}{e^x\left(x+\frac{2}{x}\right)}$

When (As) $x \to 0$, the function h take the indeterminate form $\left(\frac{\infty}{\infty}\right)$. L'Hospital's rule yields

$h_1'(x) = \frac{2}{x(x^2+1)}$ and $h_2'(x) = e^x\left(\frac{x^3+x^2+2x-2}{x^2}\right)$. So

$$\lim_{x\to 0} h(x) = \lim_{x\to 0}\frac{h_1'(x)}{h_2'(x)}$$

$$= \frac{2}{x\left(x^2+1\right)} \cdot \frac{x^2}{\left(x^3+x^2+2x-2\right)e^x} = 0.$$

2) Applying the l'Hôspital rule two times (twice) succecively, we get the limit is equal to (-2).

3) Applying the l'Hôspital rule one time, we get the limit is equal to $\left(-\frac{e}{2}\right)$.

4) Applying the l'Hôspital rule four times succecively, we get the limit is equal to $\left(\frac{1}{2}\right)$.

Chapter 6

Usual functions

6.1 Trigonometric functions

6.1.1 Sine and cosine functions

The functions $x \mapsto \sin(x)$ and $x \mapsto \cos(x)$ are defined on $\mathbb{R}$ and range $[-1, 1]$, they are $2\pi - periodics$. The function $x \mapsto \sin(x)$ is odd and $x \mapsto \cos(x)$ is even.

6.1.1.1 Derivatives:

$$\forall x \in \mathbb{R}, \quad (\sin(x))' = \cos(x) \ \mathit{and} \ (\cos(x))' = -\sin(x).$$

6.1.1.2 Variations tables

Since the two functions are periodic of their period 2π, then we restrict their domain on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.
On the other hand, since $\sin(\cdot)$ (the sine function) is odd and $\cos(\cdot)$(the cosine function) is even. we give their variation table over $\left]0, \frac{\pi}{2}\right[$.
Since $\forall x \in \left]0, \frac{\pi}{2}\right[, \ \cos(x) > 0$, then the function sine is continuous and deccreasing $\left]0, \frac{\pi}{2}\right[$ and hence, has an inverse function that is continuous and decreasing. The inverse function has domain $[-1, 1]$ and range $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and since $\forall x \in \left]0, \frac{\pi}{2}\right[, \ -\sin(x) < 0$, then the function cosine is continuous and increasing and hence, has an inverse function that is continuous and increasing $\left]0, \frac{\pi}{2}\right[$. The inverse function has domain $[-1, 1]$ and range $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

x	0		π/2
cos'x	0	−	−1
cosx	1	→	0

x	0		π/2
sin'x	1	+	0
sinx	0	→	1

Figure 6.1.2: Graph of sine

6.1.1.3 Graphs

The function "cosine" is defined on $\mathbb{R}$, even and $2\pi-$periodic.

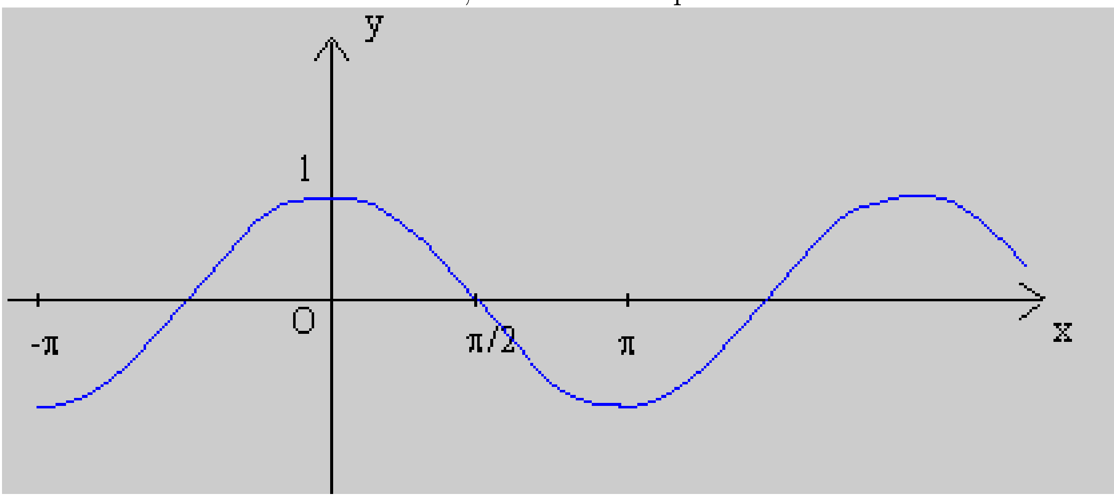
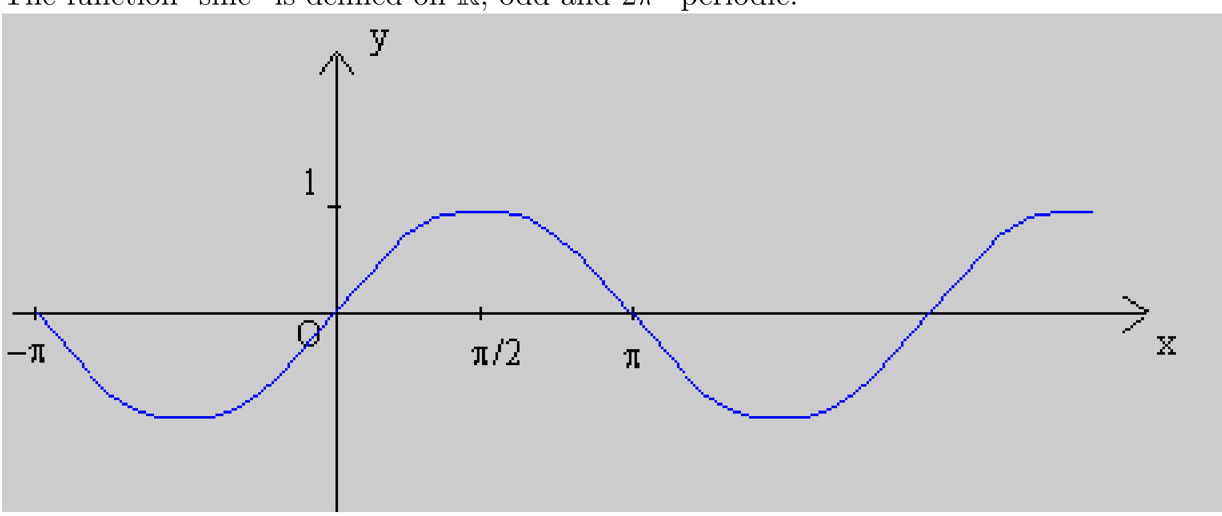


Figure 6.1.1: La courbe représentative du cosinus

The function "sine" is defined on $\mathbb{R}$, odd and $2\pi-$periodic.



6.1.2 Tangent and cotangent functions

The functions $x \mapsto \tan(x)$ and $x \mapsto \cot g(x)$ are defined on $\mathbb{R} - \left\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$ and $\mathbb{R} - \{k\pi,\ k \in \mathbb{Z}\}$ respectively, they are $\pi - \textit{periodic}$ and odds.

6.1.2.1 Derivatives:

$$\forall x \in \mathbb{R} - \left\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}, \quad (\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x),$$

et

$$\forall x \in \mathbb{R} - \{k\pi,\ k \in \mathbb{Z}\}, \quad (\cot g\,(x))' = \frac{-1}{\sin^2(x)} = -\left(1 + \cot g^2(x)\right).$$

Then, the function tangent is strictly increasing on $\mathbb{R} - \left\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$ and the function cotangent is strictly decreasing $\mathbb{R} - \{k\pi,\ k \in \mathbb{Z}\}$.

6.1.2.2 Variation tables

The two functions are periodics of period π, then we can restreint the domain of the study to a interval of longer π, for example $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ for the function tangent and $]0, \pi[$ for the function cotangent.
On the other hand, since vthe two functions are odds, we give the variation table in $\left]0, \frac{\pi}{2}\right[$.

x	0		π/2
tan'x	1	+	+∞
tanx	0	↗	∞

x	0		π/2
cotan'x	−∞	−	-1
cotanx	+∞	↘	0

6.1.2.3 Graphs

The function tangent is defined on $\mathbb{R} - \left\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$, odd and $\pi-$periodic.

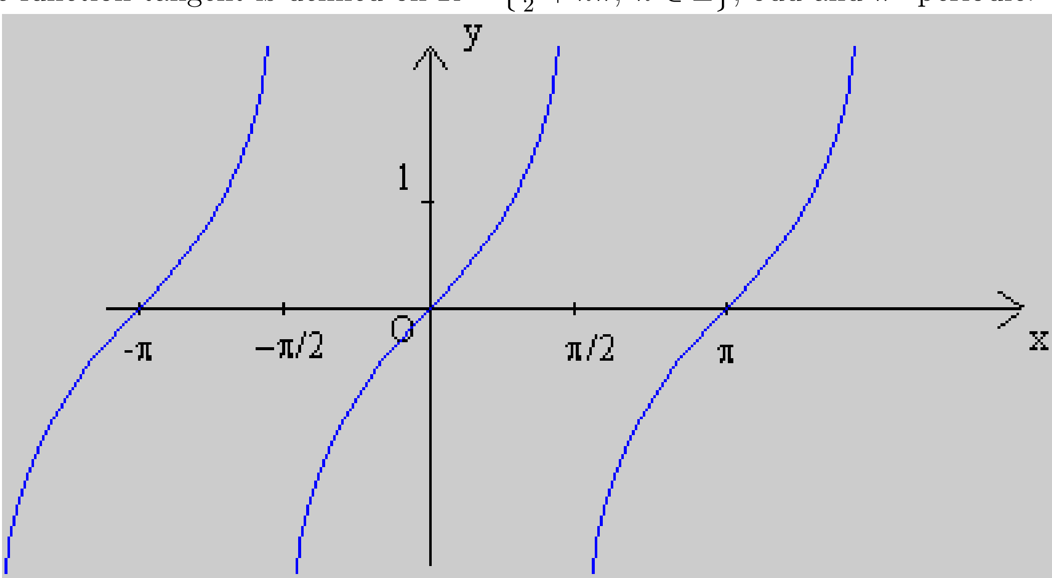


Figure 6.1.3: Graph of tangent

La fonction tangente est définie sur $\mathbb{R} - \{k\pi,\ k \in \mathbb{Z}\}$ et $\pi-$périodique.

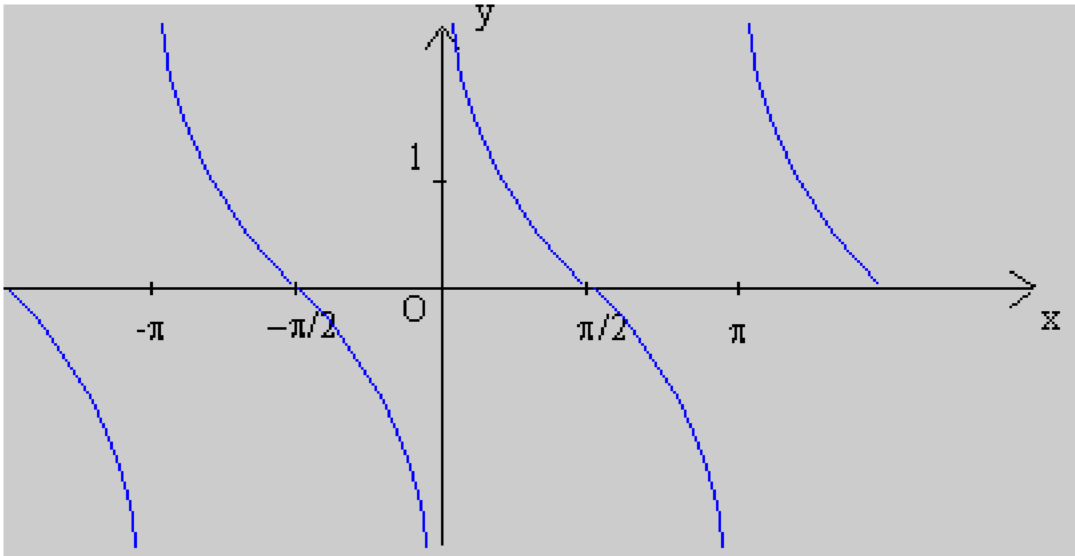


Figure 6.1.4: Graph of cotangent

6.1.3 Trigonometric formulas

6.1.3.1 Symmetry of sine and cosine

$\cos(-a) = \cos(a)\,,\ \sin(-a) = -\sin(a)\,,\ \tan(-a) = -\tan(a)$
$\cos(a+\pi) = -\cos(a)\,,\ \sin(a+\pi) = -\sin(a)\,,\ \tan(a+\pi) = \tan(a)$
$\cos(\pi-a) = -\cos(a)\,,\ \sin(\pi-a) = \sin(a)\,,\ \tan(\pi-a) = -\tan(a)$
$\cos\left(a+\frac{\pi}{2}\right) = -\sin(a)\,,\ \sin\left(a+\frac{\pi}{2}\right) = \cos(a)\,,\ \tan\left(a+\frac{\pi}{2}\right) = -\frac{1}{\tan(a)}$
$\cos\left(\frac{\pi}{2}-a\right) = \sin(a)\,,\ \sin\left(\frac{\pi}{2}-a\right) = \cos(a)\,,\ \tan\left(\frac{\pi}{2}-a\right) = \frac{1}{\tan(a)}.$

6.1.3.2 Addition formulas

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$
$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$
$\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$
$\tan(a-b) = \frac{\tan(a)-\tan(b)}{1+\tan(a)\tan(b)}.$

6.1.3.3 Double angle formulas

We take $a = b$ in the addition formulas, we obtain
$\cos(2a) = \cos^2(a) - \sin^2(a)\,,\ \sin(2a) = 2\sin(a)\cos(a)\,,\ \tan(2a) = \frac{2\tan(a)}{1-\tan^2(a)}$
The first formula $\cos^2(a) + \sin^2(a) = 1$ gives

$$\cos^2(a) = \frac{1+\cos(2a)}{2}, \quad \sin^2(a) = \frac{1-\cos(2a)}{2}.$$

6.1.3.4 Transformation of product to addition

The addition formulas gives :

$$\cos(a)\cos(b) = \frac{\cos(a-b)+\cos(a+b)}{2}, \quad \sin(a)\sin(b) = \frac{\cos(a-b)-\cos(a+b)}{2}.$$

6.2 Inverse trigonometric functions

6.2.1 Arcsine function

We would like to find the inverse function which takes y and returns to us a unique x-value so that $y = \sin(x)\,,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let $f\ :\ x \mapsto \sin(x)$. We restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
Since f is continuous, strictly increasing from $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to $f\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) = [-1, 1]$, then it is bijection. Therefore it has (admit) f^{-1} is called " Arcsine", noted $''$arcsin$''$ and we have

$$f^{-1}\ :\ [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

which is continuous, strictly increasing on $[-1, 1]$. We have $\forall x \in [-1,1]\,, \quad y = f^{-1}(x)$ and such that $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x = \sin(y) \Longleftrightarrow y = \arcsin(x)$.

6.2.1.1 Derivative

Let $f^{-1}(x) = \arcsin(x)$. We remember that f^{-1} is a inverse function of f differentiale and $\left(f^{-1}\right)'(x) = \frac{1}{f'(y)}$ with $y = f^{-1}(x)$ *et* $f'(y) \neq 0$. Thus

$$\begin{cases} x = \sin(y) \\ x \in [-1,1] \end{cases} \Longleftrightarrow \begin{cases} y = \arcsin(x)\,, \\ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \end{cases}$$

The function f is differentiable on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and it's derivative is defined by $f'(y) = \cos(y) \neq 0$, $\forall y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and on the other hand, the function f^{-1} is a differentiable on $]-1,1[$. Thus

$$\left(f^{-1}\right)'(x) = (\arcsin(x))' = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}, \ |x| < 1.$$

6.2.1.2 Table of variation

x	0		1
Arcsin'x	1	+	$+\infty$ ||
Arcsin x	0	$\nearrow$	$\pi/2$

6.2.1.3 Graph

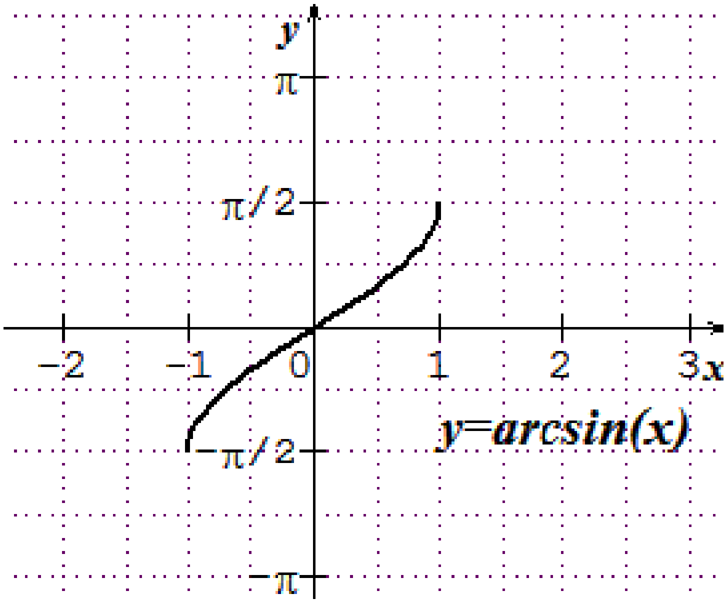


Figure 6.2.1: Graph of arcsinus

Exercise 27. Show that if $0 < x < 1$, then $\arcsin(x) < \frac{x}{\sqrt{1-x^2}}$.
Solution: Using the mean value theorem on the function $t \mapsto \arcsin(t)$ on $[0, x]$, $0 < x < 1$, hence $\exists c \in]0, x[$ such that

$$\arcsin(x) - \arcsin(0) = x\frac{1}{\sqrt{1-c^2}}.$$

We know that $c \in]0, x[$ and the function $x \longmapsto \frac{1}{\sqrt{1-x^2}}$ is strictly increasing on $]0, 1[$, then

$$\frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-x^2}}.$$

Therefore, $\arcsin(x) < \frac{x}{\sqrt{1-x^2}}, \quad if\ 0 < x < 1.$

6.2.2 Arccosine function

The definition for "arccos" is developed in the same way.

Let $g : x \mapsto \cos(x)$. We restrict the domain to $[0, \pi]$.
Since g is continuous, strictly decreasing from $[0, \pi]$ to $g([0, \pi]) = [-1, 1]$, then it is bijection. Thus, it has (admit) a inverse function g^{-1} called " arccosinus", denoted by "arccos" and we have

$$f^{-1} : [-1, 1] \to [0, \pi],$$

which is continuous and strictly decreasing on $[-1, 1]$.

$\forall x \in [-1, 1], \quad y = g^{-1}(x)$ and such that $y \in [0, \pi]$ et $x = \cos(y) \Longleftrightarrow y = \arccos(x)$.

6.2.2.1 Derivative

Let $g^{-1}(x) = \arccos(x)$. We have

$$\begin{cases} x = \cos(y) \\ x \in [-1, 1] \end{cases} \Longleftrightarrow \begin{cases} y = \arccos(x), \\ y \in [0, \pi]. \end{cases}$$

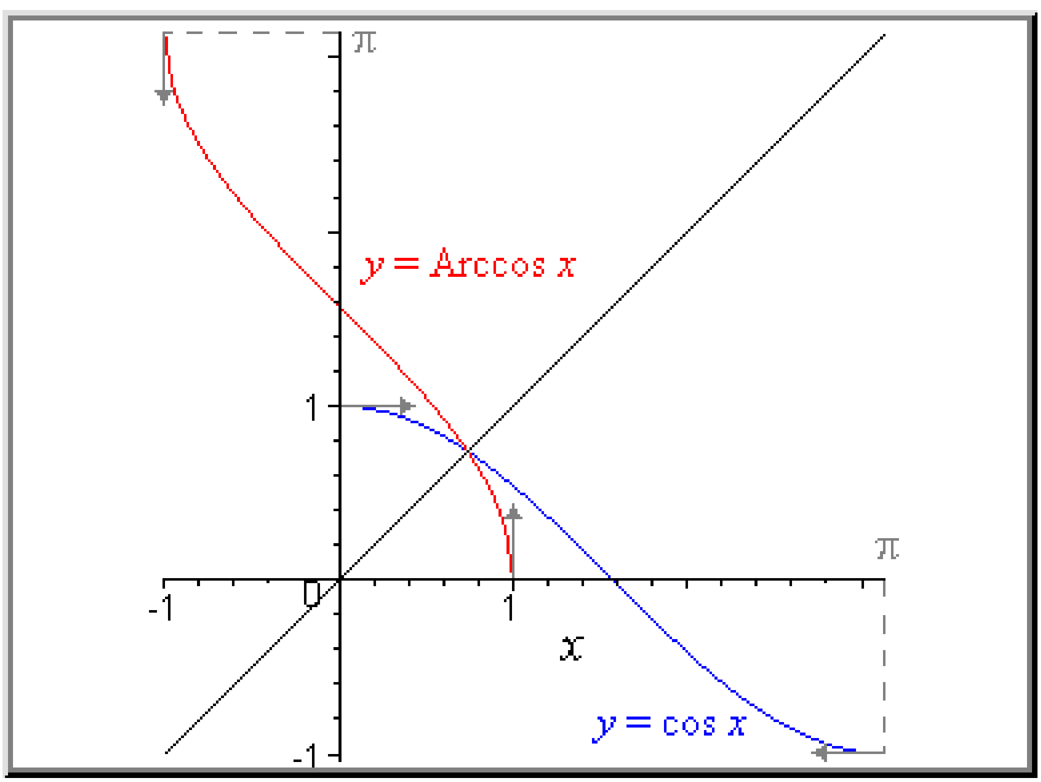
The function is differentiable on $g\ ]0, \pi[$ and it's derivative defined by $g'(y) = -\sin(y) \neq 0,\ \forall y \in\ ]0, \pi[$ and on the other hand, the function g^{-1} is differentiable on $]-1, 1[$, hence

$$\left(g^{-1}\right)'(x) = (\arccos(x))' = -\frac{1}{\sin(y)} = -\frac{1}{\sqrt{1-\cos^2(y)}} = \frac{-1}{\sqrt{1-x^2}},\ |x| < 1.$$

6.2.2.2 Table of variation

x	-1	1
$\arccos x$	π ↘	0

6.2.2.3 Graph



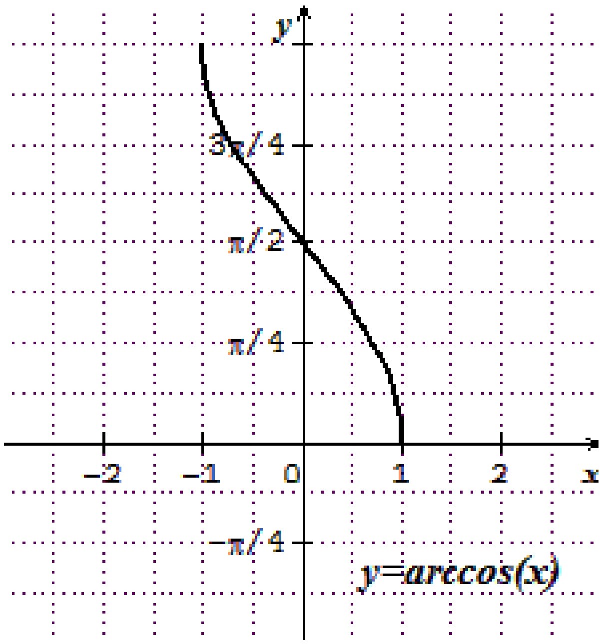


Figure 6.2.2: Graph of arccosinus

Exercise 28. Calculate the definition domain and the set which the function f is differentiable and calculate their derivative:
$f(x) = \arcsin\left(\frac{1-x^2}{1+x^2}\right)$.

Solution:

- Definition domain:
$$D_f = \left\{x/x \in \mathbb{R} \,:\, \left|\frac{1-x^2}{1+x^2}\right| \leq 1\right\} = \mathbb{R}.$$

- The function f is composite of two functions $u : x \longmapsto \arcsin(x)$ and $v : x \longmapsto \frac{1-x^2}{1+x^2}$ with u is differentiable on $]-1, 1[$. Then, f is differentiable on
$$\mathbb{R} - \left\{x/x \in \mathbb{R} \,:\, \left|\frac{1-x^2}{1+x^2}\right| = 1\right\} = \mathbb{R}^*,$$

and their derivative isdefined by
$$f'(x) = \frac{v'}{\sqrt{1-v^2}} = \frac{-2x}{|x|(1+x^2)}, \quad \forall x \neq 0.$$

6.2.3 Arctangent function

Also, the definitions for "arctan" is developed in the same way.

Let the function $f : x \mapsto \tan(x)$ on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

Since f is continuous, strictly decreasing from $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ to $f\left(\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)=\mathbb{R}$, then it is bijection. Thus, it has (admit) a inverse function f^{-1} called " arctangent", denoted "arctan" and we have

$$f^{-1}\,:\,\mathbb{R}\rightarrow\left]-\frac{\pi}{2},\frac{\pi}{2}\right[,$$

is also continuous and strictly decreasing on $\mathbb{R}$.

$\forall x\in\mathbb{R},\quad y=f^{-1}\left(x\right)$ and such that $y\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ and $x=\tan\left(y\right)\Longleftrightarrow y=\arctan\left(x\right)$.

6.2.3.1 Derivative

Let $f^{-1}\left(x\right)=\arctan\left(x\right)$. We have

$$\left\{\begin{array}{c} x=\tan\left(y\right)\\ x\in\mathbb{R}\end{array}\right.\quad\Longleftrightarrow\left\{\begin{array}{c} y=\arctan\left(x\right)\\ y\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[.\end{array}\right.$$

On the other hand, the function f is differentiable on $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ and their derivative defined by $f'\left(y\right)=1+\tan^{2}\left(y\right)\neq0$. Further, the function f^{-1} is differentiable on $\mathbb{R}$. So, we have

$$\left(f^{-1}\right)'\left(x\right)=\left(\arctan\left(x\right)\right)'=\frac{1}{1+\tan^{2}\left(y\right)}=\frac{1}{1+x^{2}},\ \forall x\in\mathbb{R}.$$

6.2.3.2 Variation table

x	$-\infty$		$+\infty$
arctan x	$-\frac{\pi}{2}$	$\nearrow$	$\frac{\pi}{2}$

6.2.3.3 Graph

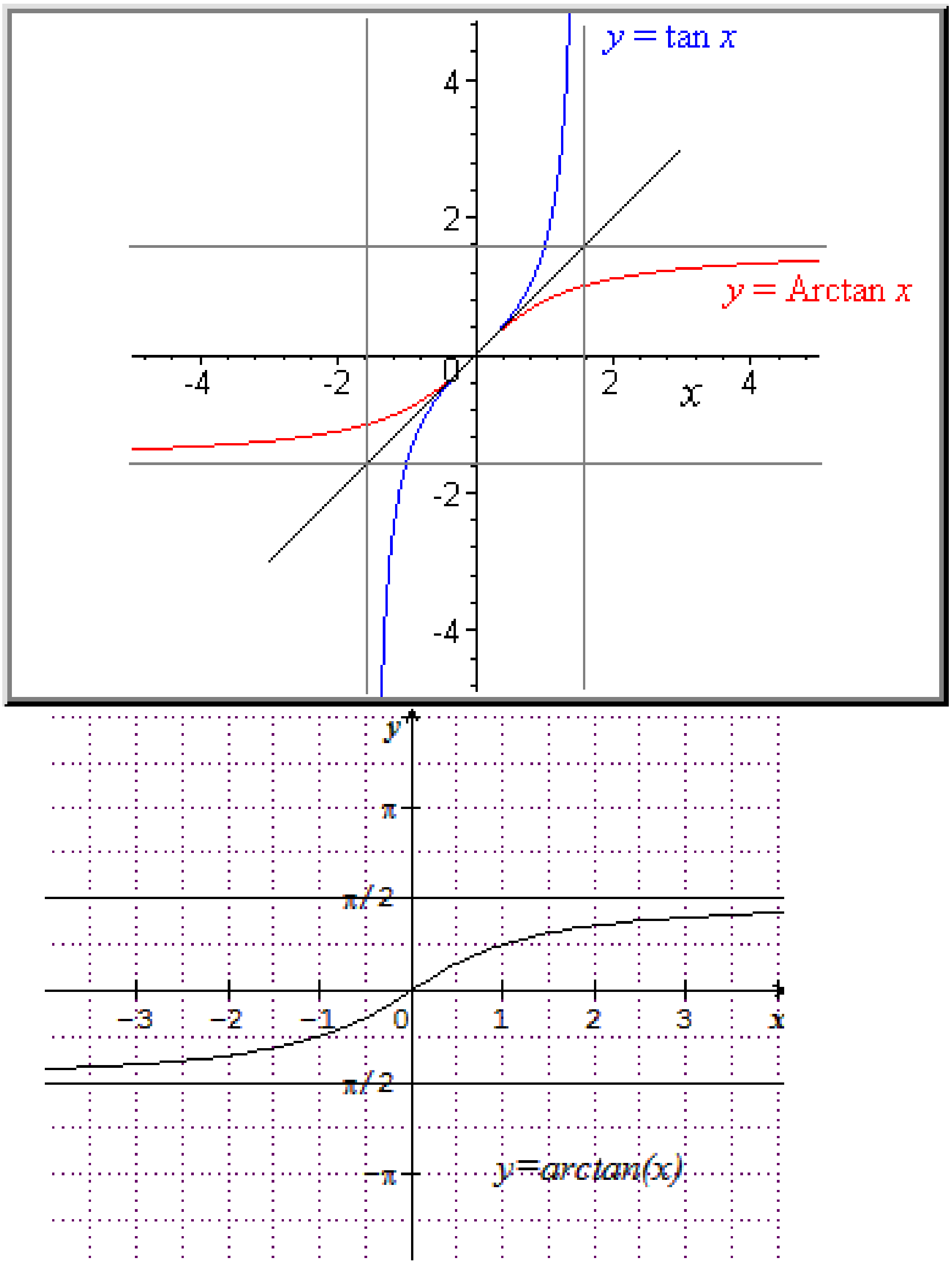


Figure 6.2.3: Graph of arctan

Exercise 29. Determinate the definition domain and the set which the function f is differentiable and calculate their derivative:

1) $f(x) = \frac{e^{\arctan(x)}}{1+\sqrt{x}}$.

2) $g(x) = \arctan\left(\frac{1-x}{1+x}\right)$.

Solution:

- 1) Definition domain:

$$D_f = \{x/x \in \mathbb{R} \,:\, x \geq 0\} = \mathbb{R}^+ = [0, +\infty[\,.$$

- The function f is differentiable on (over) $]0, +\infty[$ and they derivative defined by

$$f'(x) = \frac{e^{\arctan(x)}\left[(1+\sqrt{x})(2\sqrt{x}) - 1 - x^2\right]}{2(1+x^2)(1+\sqrt{x})^2\sqrt{x}}.$$

- 2) Definition domain:

$$D_g = \{x/x \in \mathbb{R} \,:\, x \neq -1\} = \mathbb{R} - \{-1\}\,.$$

The function g is differentiable on $\mathbb{R} - \{-1\}$ and they derivative defined by

$$g'(x) = \frac{-2}{\left[(1+x)^2 + (1-x)^2\right]}$$

6.2.4 Arccotangent function

The function cotangent defined over $]0, \pi[$, we define the inverse function ”arccotangent” (denoted: arccotan), continuous and strictly inceasing from $\mathbb{R}$ to $]0, \pi[$. It's differentiable and we have

$$\left(f^{-1}\right)'(x) = (arccotan\,(x))' = \frac{-1}{1+x^2},\ \forall x \in \mathbb{R}.$$

6.2.4.1 Variation table

x	$-\infty$		$+\infty$
Arccotan'x	0^-	$-$	0^-
Arccotan x	π	$\searrow$	0

6.2.5 Properties

1) $\forall x \in [-1, 1]\,,\quad \arcsin(x) + \arccos(x) = \frac{\pi}{2}.$
2) $\forall x \in [-1, 1]\,,\quad \arccos(x) + \arccos(-x) = \pi.$

3) $\arctan(x) + \arctan\left(\frac{1}{x}\right) = \begin{cases} -\frac{\pi}{2}, & if\ x < 0, \\ \frac{\pi}{2}, & if\ x > 0. \end{cases}$

4) $\forall \alpha,\ \beta \in \mathbb{R}$ such that $\alpha\beta > 0$, we have

$$\arctan(\alpha) + \arctan(\beta) = \arctan\left(\frac{\alpha+\beta}{1-\alpha\beta}\right),\ \alpha\beta \neq 1.$$

5) $\forall \alpha,\ \beta \in \mathbb{R}$ such that $\alpha\beta > 0$, we have

$$\forall \alpha\beta \neq 0, \qquad \arctan(\alpha) + \arctan(\beta) - \arctan\left(\frac{\alpha+\beta}{1-\alpha\beta}\right) = \begin{cases} \pi, & if\ \alpha > 0, \\ -\pi, & if\ \alpha < 0. \end{cases}$$

6) $\forall x \in \mathbb{R},\ \arctan(x) + arccotan(x) = \frac{\pi}{2}$.

Proof. 1) Wewill prove that $\forall x \in [-1,1], \quad \arcsin(x) + \arccos(x) = \frac{\pi}{2}$. □

We say that

$$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin(\alpha),$$

and

$$-\frac{\pi}{2} \leq \arcsin(x) \leq \frac{\pi}{2} \Longrightarrow 0 \leq \frac{\pi}{2} - \arcsin(x) \leq \pi.$$

So,

$$\cos\left(\frac{\pi}{2} - \arcsin(x)\right) = \sin(\arcsin(x)) = x.$$

Therefore

$$\arccos\left[\cos\left(\frac{\pi}{2} - \arcsin(x)\right)\right] = \arccos(x) \Longrightarrow \frac{\pi}{2} - \arcsin(x) = \arccos(x)$$

$$\Longrightarrow \frac{\pi}{2} = \arcsin(x) + \arccos(x).$$

2) We will prove that $\forall x \in [-1,1], \quad \arccos(x) + \arccos(-x) = \pi$.
We say that

$$\cos(\pi - \alpha) = -\cos(\alpha),$$

and

$$0 \leq \arccos(x) \leq \pi \Longrightarrow 0 \leq \pi - \arccos(x) \leq \pi.$$

So,

$$\cos(\pi - \arccos(x)) = -\cos(\arccos(x)) = -x.$$

Therefore

$$\arccos[\cos(\pi - \arccos(x))] = \arccos(-x) \Longrightarrow \pi - \arccos(x) = \arccos(-x)$$

$$\Longrightarrow \pi = \arccos(x) + \arccos(-x).$$

3) We will prove that

$$f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right) = \begin{cases} -\frac{\pi}{2}, & si\ x < 0, \\ \frac{\pi}{2}, & si\ x > 0. \end{cases}$$

We have, $\forall x \in \mathbb{R}^*,\ f'(x) = \frac{1}{1+x^2} - \frac{\frac{1}{x^2}}{1+\frac{1}{x^2}} = 0 \Rightarrow f(x) = \begin{cases} c, & si\ x > 0, \\ d, & si\ x < 0, \end{cases}$

so, for $x = 1$, we have

$$f(1) = \arctan(1) + \arctan(1) = \frac{\pi}{2}, \quad si\ x > 0,$$

for $x=-1$, we have

$$f(-1)=\arctan(-1)+\arctan(-1)=-\frac{\pi}{2},\quad si\ x<0.$$

Then

$$f(x)=\arctan(x)+\arctan\left(\frac{1}{x}\right)=\begin{cases}-\frac{\pi}{2}, & if\ x<0,\\ \frac{\pi}{2}, & if\ \ x>0.\end{cases}$$

Exercise 30. Prove that
1) $\forall x\in[-1,1],\quad \arcsin(x)+\arcsin(-x)=0.$

2)$\forall\alpha,\ \beta\in\mathbb{R}$ such that $\alpha\beta>0$, we have

$$\frac{\beta-\alpha}{1+\beta^2}<\arctan(\alpha)-\arctan(\beta)<\frac{\beta-\alpha}{1+\alpha^2}.$$

Solution: 1) Since the function $x\longmapsto\arcsin(x)$ is odd, then we have

$$\forall x\in[-1,1],\quad \arcsin(x)+\arcsin(-x)=0.$$

2) Applying the mean value theorem to the function $f:x\mapsto\arctan(x)$ on the interval $[\alpha,\beta]$.

Exercise 31. Prove that $\forall x\in[-1,1]$, we have
1) $\cos(\arcsin(x))=\sqrt{1-x^2}$.
2) $\sin(\arccos(x))=\sqrt{1-x^2}$.
3) $\cos(2\arctan(x))=\frac{1-x^2}{1+x^2}$.
4) $\sin(2\arctan(x))=\frac{2x}{1+x^2}$.
5) Deduce $\sin(\arccos(x)+2\arctan(y))=\frac{\left(1-y^2\right)\sqrt{1-x^2}+2xy}{1+y^2}$.
6) Determinate the definition domain of the following function :

$$f(x)=\arccos\left(1-2x^2\right)$$

Solution: 1) Let $x\in\left]-1,1\right[$, we set $y=\arcsin(x)\Rightarrow x=\sin(y),\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
So, since $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ then $\cos(y)\geq0$.
therefore, we have

$$\cos(\arcsin(x))=\cos(y)=\pm\sqrt{1-\sin^2(y)}\Rightarrow\cos(y)=\sqrt{1-\sin^2(y)}\ car\ \cos(y)\geq0$$

$$\Rightarrow\cos(\arcsin(x))=\sqrt{1-x^2}.$$

2) By the same way (similarly we get the result).
3) The expression is defined for $x\in\mathbb{R}$. Further, we say that $\cos(2\theta)=\frac{1-\tan^2(\theta)}{1+\tan^2(\theta)}$. So, we have

$$\cos(2\arctan(x))=\frac{1-\tan^2(\arctan(x))}{1+\tan^2(\arctan(x))}=\frac{1-x^2}{1+x^2}.$$

4) The expression is defined for $x \in \mathbb{R}$. Further, we say that $\sin(2\theta) = \frac{2\tan(\theta)}{1+\tan^2(\theta)}$. So, we have

$$\cos(2\arctan(x)) = \frac{1-\tan(\arctan(x))}{1+\tan^2(\arctan(x))} = \frac{1-x^2}{1+x^2}$$

5) The expression is defined for $x \in [-1,1]$ et $y \in \mathbb{R}$. Further, we say that $\arccos(x) \in [0,\pi]$, $\arctan(y) \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

So, we obtain the following expression

$$\sin(\arccos(x) + 2\arctan(y)) = \frac{(1-y^2)\sqrt{1-x^2}+2xy}{1+y^2}.$$

6) Since $1-2x^2 \in [-1,1]$ then $|x| \leq 1$. Thus, $D_f = [-1,1]$.

Using the chain rule with Inverse trigonometric functions: Now let's see how to use the chain rule to find the derivatives of inverse trigonometric functions with more interesting functional arguments.

Exercise 32. Calculate vthe derivative of the following functions:

1) $f(x) = \arcsin(3x-1)$.
2) $f(x) = \arcsin\left(\frac{x^2-1}{x^2}\right)$.
3) $f(x) = \arccos\left(\frac{1}{1-x}\right)$.
4) $f(x) = \arctan\left(\frac{x}{2+x^2}\right)$.
5) $f(x) = arc\cot\left(e^{2x}\right)$.
6) $f(x) = e^{\arcsin(x)}$.
7) $f(x) = \arcsin\left(\sqrt[3]{x}\right)$.
8) $f(x) = \frac{e^{\arctan(x)}}{(1+x)^{\frac{3}{2}}}$.

The reader is left to solve the exercice (see, exercices 28 and 29).

Chapter 7

Hyperbolic and inverse hyperbolic functions

7.1 Hyperbolic functions

The chapter begins with a review of the hyperbolic functions and its inverse that are studied in exponontial courses. We define the four main hyperbolic functions, and sketch their graphs. We also discuss some identities relating these functions, and mention their inverse functions and reciprocal functions. We next apply methods of calculus to obtain formulas for derivatives. The hyperbolic functions are used in the physical sciences and engineering to describe the shape of a Oexible cablc that is supported at each end, to find the velocity of an object in a resisting medium such as air or water, and to study the diffusion of radon gas through a basement wall. The chapter closes with a discussion of the inverse hyperbolic functions. These functions are used primarily for evaluating certain types of integrals

Definition 33. We define hyperbolic sine function (pronounced as "sinh"), hyperbolic cosine function (pronounced as "cosh"), hyperbolic tangent (tanh) and hyperbolic cotangent (coth) as follows :

$$\forall x \in \mathbb{R},\ \cosh(x) = \frac{e^{x}+e^{-x}}{2},\ \sinh(x) = \frac{e^{x}-e^{-x}}{2},\ \tanh(x) = \frac{sh(x)}{ch(x)},$$

and

$$\forall x \in \mathbb{R}^{*}, \coth(x) = \frac{1}{\tanh(x)}.$$

Note that, we can pronounced respectively the above functions as : *sh*, *ch*, *th*, *and cth.*

Remark 17. Note that

1) $\sinh(x) = 0 \Longleftrightarrow x = 0$.
2) $\cosh(x) > 0 \Longleftrightarrow x \in \mathbb{R}$.
3) $\forall x \in \mathbb{R},\ \tanh(x) = \frac{e^{2x}-1}{e^{2x}+1}$.

Modeling : 1) The hyperbolic cosine function can be used to describe the shape of a uniform flexible cable or chain, whose ends are supported from the same height. The shapc

of the cablc appears to be a parabola, but is actually a catenary (after the Latin word for chain). If we introduce a coordinate system, it can be shown that an equation corresponding to the shape of the cable is $y = a\cosh\left(\frac{x}{a}\right)$ for some real number a. The hyperbolic cosine function also occurs in the analysis of motion x in a resisting medium. If an object is dropped from a given height and if air resistance is disregarded, then the y distance that it falls in t seconds is $y = \frac{1}{2}gt^2$, where g is a gravitational constant. However, air resistance cannot always be disregarded. As the velocity of the object increases, air resistance may significantly affect its motion. For example, if the air resistance is directly proportional to the square of the velocity, then the distance y that the object falls in t seconds is given by $y = A\ln(\cosh(Bt))$ for constants A and B.
2) Radon gas can readily diffuse through solid materials such as brick and cement. If the direction of diffusion in a basement wall is perpend icular to the surface, then the radon concentration $f(x)$ $(in\ jouls/cm^3\ )$ in the air-filled pores within the wall at a distance x from the outside surface can be approximated by $f(x) = A\sinh(qx) + B\cosh(qx) + k$, where the constant q depends on the porosity of the wall, the half-life of radon, and a diffusion coefficient: the constant k is the maximum radon concentration in the air-filled pores; and A and B are constants that depend on initial conditions.

7.1.1 Algebric properties

1) $\sinh(x) + \cosh(x) = e^x$
2) $\cosh(x) - \sinh(x) = e^{-x}$,
3) $\cosh^2(x) - \sinh^2(x) = 1$,
4) $1 - \tanh^2(x) = \frac{1}{\cosh^2(x)}$.

7.1.2 Derivatives

Derivative formulas for the hyperbolic functions are listed in the following proposition.

Proposition 21. *The functions $f : x \mapsto \sinh(x)$, $g : x \mapsto \cosh(x)$ et , $h : x \mapsto \tanh(x)$ are differentiable on $\mathbb{R}$ and they derivatives are :*

$$\forall x \in \mathbb{R},\ f' : x \mapsto \cosh(x),$$
$$\forall x \in \mathbb{R},\ g' : x \mapsto \sinh(x),$$

and to differentiate $h(x) = \tanh(x)$, we apply the quotient rule, we obtain:

$$\forall x \in \mathbb{R},\ h' : x \mapsto 1 - \tanh^2(x).$$

7.1.3 Sens of variation

1) $\forall x \in \mathbb{R},\ f'(x) = \cosh(x) \geq 1 > 0$. So, the function f is strictly increasing on $\mathbb{R}$.
2) $\forall x \in \mathbb{R}_+^*,\ g'(x) = \sinh(x) > 0$ and $\forall x \in \mathbb{R}_-^*,\ g'(x) = \sinh(x) < 0$. So, the function g is strictly increasing on $\mathbb{R}_+^*$ is strictly decreasing on $\mathbb{R}_-^*$.
3) $\forall x \in \mathbb{R},\ h' : x \mapsto 1 - \tanh^2(x) = \frac{1}{ch^2 x} > 0$. So, the function h is strictly increasing on $\mathbb{R}$.

7.1.4 Parity

1) The function $f : x \mapsto \cosh(x)$ even function.
2) The functions $g : x \mapsto \sinh(x)$ and $h : x \mapsto \tanh(x)$ are odd.

7.1.5 Variation tables

The functions f, g are even and odd, respectively, then we can restrict the domain of study on the interval $[0, +\infty[$.

x	0		$+\infty$
ch'x	0	+	$+\infty$
ch x	1	$\nearrow$	$+\infty$

x	0		$+\infty$
sh'x	1	+	$+\infty$
sh x	0	$\nearrow$	$+\infty$

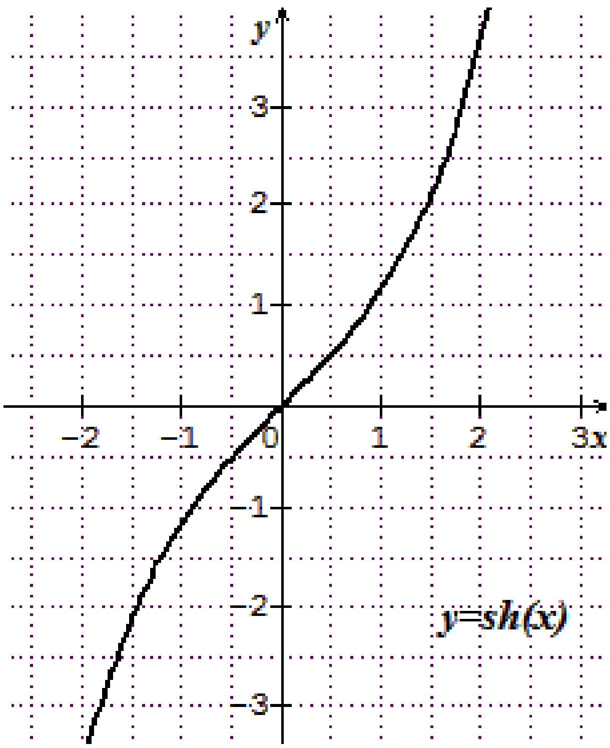
x	0		$+\infty$
th'x	1	+	0
th x	0	$\nearrow$	1

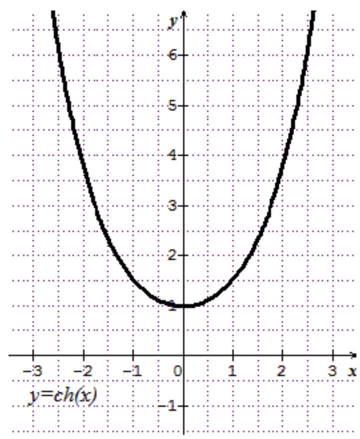
x	0		$+\infty$
coth'x	$-\infty$	−	0^-
coth x	$+\infty$	$\searrow$	1

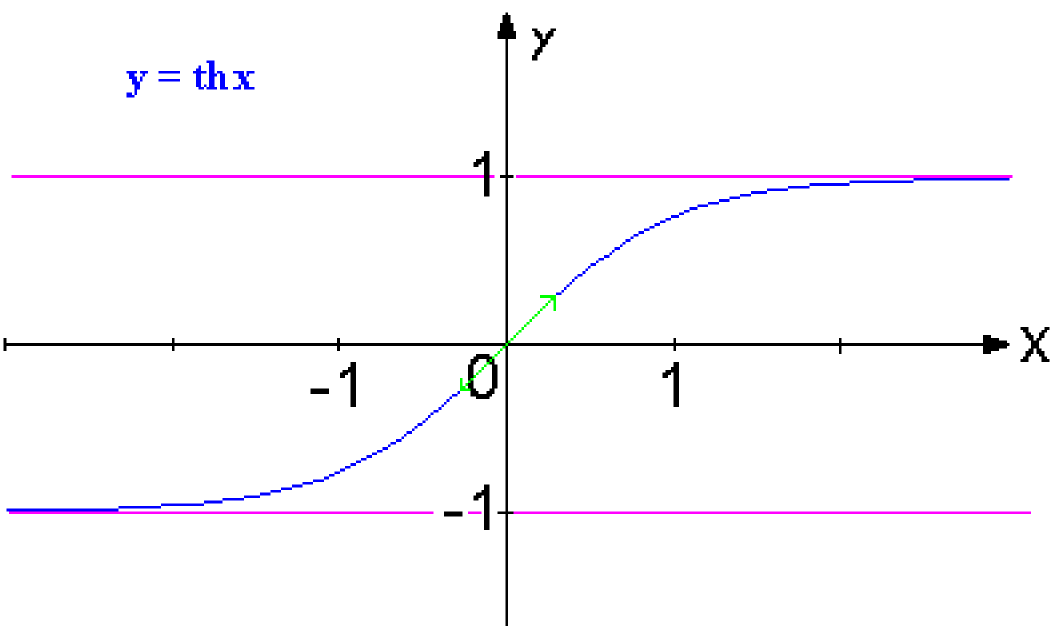
1) Since $\lim\limits_{x \to +\infty} \frac{\sinh(x)}{x} = +\infty$, then the graph of the function f admit a parabolic branch of asymptotic direction oy.
2) Since $\lim\limits_{x \to +\infty} \frac{\cosh(x)}{x} = +\infty$, then the graph of the function g admit a parabolic branch of asymptotic direction oy.

Figure 7.1.1: Graphs of sinh and cosh

7.1.6 Graphs







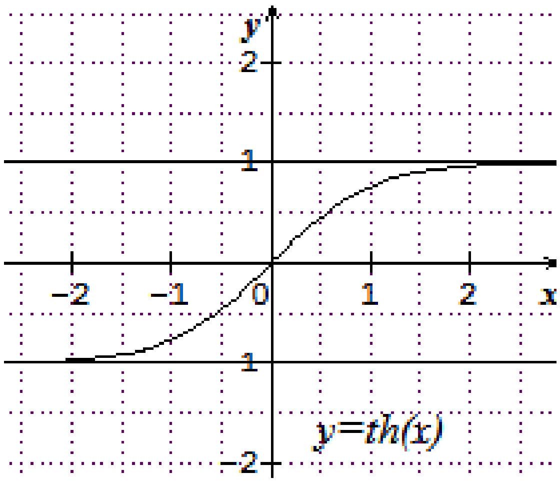


Figure 7.1.2: Graph of tanh

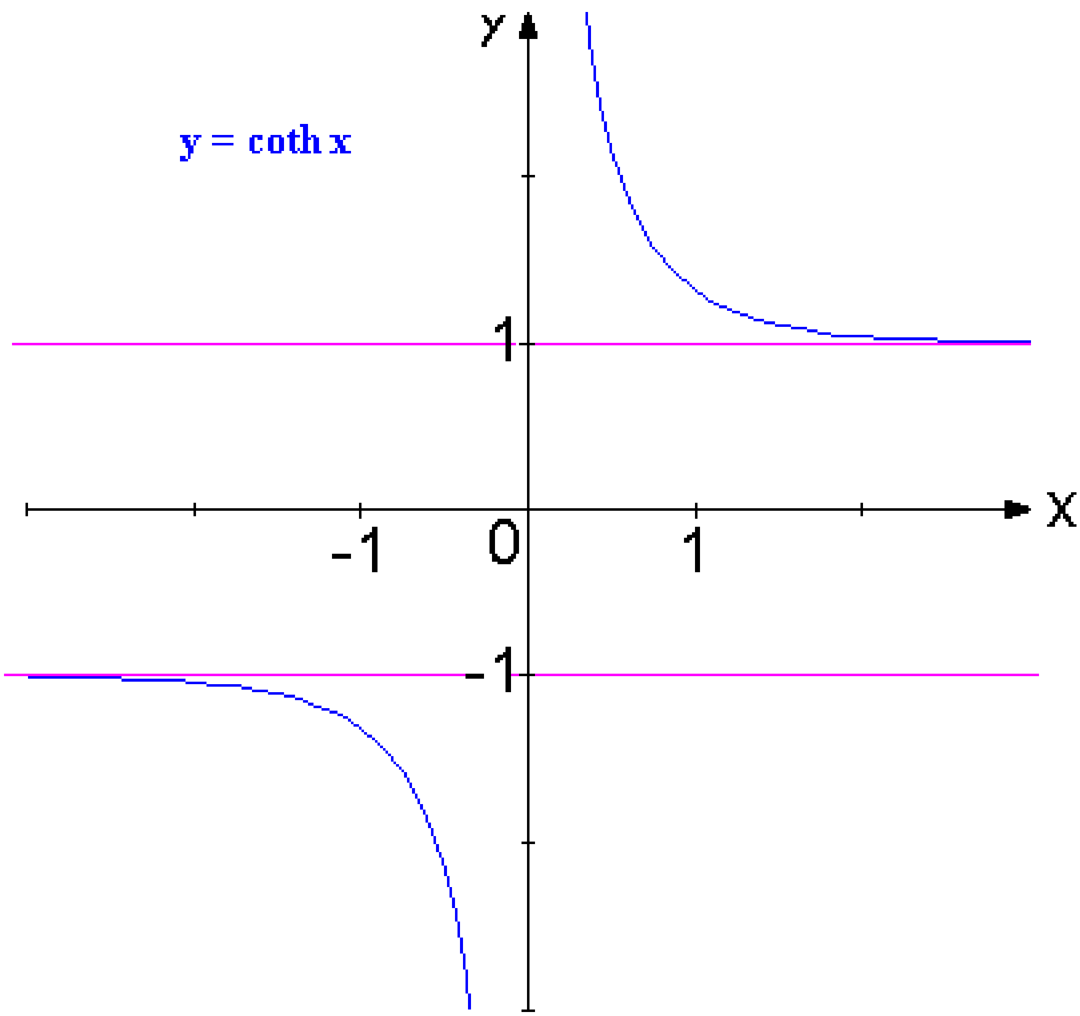


Figure 7.1.3: Graph of coth

7.1.7 Identities for hyperbolic functions

7.1.7.1 Basic formulas

$\cosh(x)+\sinh(x)=e^{x}, \quad \cosh(x)-\sinh(x)=e^{-x},$
$\cosh^{2}(x)-\sinh^{2}(x)=1, \quad 1-\tanh^{2}(x)=\frac{1}{\cosh^{2}(x)}.$

7.1.7.2 Symmetry

$\cosh(-x)=\cosh(x), \quad \sinh(-x)=-\sinh(x), \quad \tanh(-x)=-\tanh(x).$

7.1.7.3 Addition formulas

$\cosh(a+b)=\cosh(a)\cosh(b)+\sinh(a)\sinh(b)$
$\cosh(a-b)=\cosh(a)\cosh(b)-\sinh(a)\sinh(b)$
$\sinh(a+b)=\sinh(a)\cosh(b)+\cosh(a)\sinh(b)$
$\sinh(a-b)=\sinh(a)\cosh(b)-\cosh(a)\sinh(b),$
$\tanh(a+b)=\frac{\tanh(a)+\tanh(b)}{1+\tanh(a)\tanh(b)}$
$\tanh(a-b)=\frac{th(a)-th(b)}{1-th(a)th(b)}$

7.1.7.4 Double angle formulas

We take $a=b$, the formulas of addition gives
$\cosh(2a)=\cosh^{2}(a)+\sinh^{2}(a), \ \sinh(2a)=2\cosh(a)\sinh(a)$
$\tanh(2a)=\frac{2\tanh(a)}{1+\tanh^{2}(a)}.$
Using $\cosh^{2}(x)-\cosh^{2}(x)=1$, we obtain
$\cosh(2a)=2\cosh^{2}(a)-1=1+\sinh^{2}(a).$

7.1.7.5 Transformation of product to sum

$\cosh(a)\cosh(b)=\frac{\cosh(a+b)+\cosh(a-b)}{2}$
$\sinh(a)\sinh(b)=\frac{\cosh(a+b)-\cosh(a-b)}{2}$
$\sinh(a)\cosh(b)=\frac{sh(a+b)+sh(a-b)}{2}$
$\cosh(a)\sinh(b)=\frac{\sinh(a+b)-\sinh(a-b)}{2}$

Example 29. Find the linear form of $\cosh^{5}(x)$. We have

$$\cosh^{5}(x)=\left(\frac{e^{x}+e^{-x}}{2}\right)^{5},$$

We devolope this expression, we obtain

$$\cosh^{5}(x)=\frac{1}{2^{4}}\left(\cosh(5x)+5\cosh(3x)+10\cosh(x)\right).$$

Example 30. We will represente $\sinh(3x)$ via power of $\sinh(x)$. We have

$$\sinh(3x)=\frac{e^{3x}-e^{-3x}}{2},$$

and

$$(\cosh(x)+\sinh(x))^3 = e^{3x}, \quad (\cosh(x)-\sinh(x))^3 = e^{-3x}.$$

We devolope the two expressions and we simplify, we obtain :

$$\sinh(3x) = \sinh^2(x) + 3\cosh^2(x)\sinh(x) = 4\sinh^3(x) + 3\sinh(x).$$

Exercise 33. Let the function $f(x) = \sqrt{\frac{\cosh(x)-1}{\cosh(x)+1}}$.

1) Determine the definition domain of f.
2) Simplify the expression of f.
3) Deduce and simplify $argthf(x)$.

Solution: 1) The definition domain of f is $D_f = \mathbb{R}$. In fact, $\forall x \in \mathbb{R}, \quad 1+\cosh(x) \neq 0$, $\frac{\cosh(x)-1}{\cosh(x)+1} \geq 0$.

2) We have $\frac{\cosh(x)-1}{\cosh(x)+1} = \frac{e^x+e^{-x}-2}{e^x+e^{-x}+2} = \left(\frac{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}\right)^2 = \left(\frac{\sinh\left(\frac{x}{2}\right)}{\cosh\left(\frac{x}{2}\right)}\right)^2 = \tanh^2\left(\frac{x}{2}\right)$.

So, $f(x) = \sqrt{\frac{\cosh(x)-1}{\cosh(x)+1}} = \left|\tanh\left(\frac{x}{2}\right)\right|$.

Exercise 34. 3) We deduce $arg\tanh f(x) = arg\tanh\sqrt{\frac{\cosh(x)-1}{\cosh(x)+1}} = arg\tanh\left|\tanh\left(\frac{x}{2}\right)\right| = \left|\frac{x}{2}\right|$.

7.2 Inverse hyperbolic functions

Looking at the graphs of the hyperbolic functions, we see that with appropriate range restrictions, they all have inverses. Most of the necessary range restrictions can be discerned by close examination of the graphs. The domains and ranges of the inverse hyperbolic functions are summarized in the following:

7.2.1 Inverse hyperbolic sine function

Definition 34. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sinh(x)$ is continuous and strictly increasing. Thus it is bijection map. It has an reciprocal function f^{-1} called "inverce hyperbolic sine function (pronounced as$arg\sinh$) is continuous and strictely increasing on $\mathbb{R}$. We write $f^{-1} : x \mapsto arg\sinh(x)$ and we have

$$\forall x \in \mathbb{R},\ y = arg\sinh(x) \Longleftrightarrow \forall y \in \mathbb{R},\ x = \sinh(y).$$

Remark 18. 1) The function $''arg\sinh''$ is odd.

2) Since the hyperbolic sine function is defined in terms of the exponential function, we should not find it surprising that the inverse hyperbolic sine function may be expressed in terms of the natural logarithm function.

We have $\forall x \in \mathbb{R},\ arg\sinh(x) = \ln\left(x+\sqrt{x^2+1}\right)$ (logarithm expression). In fact, we have

$$\cosh^2(y) - \sinh^2(y) = 1$$

and

$$\forall x \in \mathbb{R},\ y = arg\sinh(x) \Longleftrightarrow \forall y \in \mathbb{R},\ x = \sinh(y),$$

so,

$$\cosh(y) = \sqrt{1+x^2},$$

and since $e^y = \cosh(y) + \sinh(y)$, then

$$e^y = x + \sqrt{1+x^2}.$$

Thus,

$$y = \ln\left(x + \sqrt{1-x^2}\right) \Longleftrightarrow arg\sinh(x) = \ln\left(x + \sqrt{1+x^2}\right), \quad x \in \mathbb{R}.$$

7.2.1.1 Derivative

The function "$arg\sinh$" is differentiable on $\mathbb{R}$ and its derivative is defined as

$$f'(x) = \frac{1}{\sqrt{1+x^2}}.$$

7.2.1.2 Variation table

The function "$arg\sinh$" is odd, then we can rectrict the domain of study on the interval $[0, +\infty[$.

x	0		$+\infty$
Argsh'x	1	+	0^+
Argsh x	0	$\nearrow$	$+\infty$

7.2.1.3 Graph

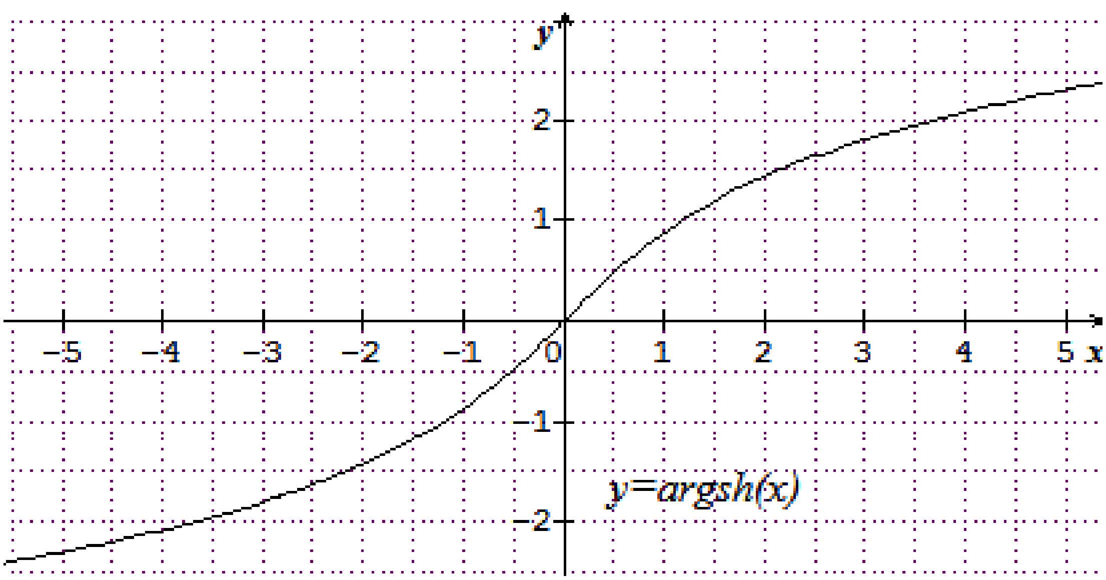


Figure 7.2.1: Graph of argsinh

7.2.2 Inverse hyperbolic cosine function

Definition 35. The function $g : [0, +\infty[\to [1, +\infty[$ defined by $g(x) = \cosh(x)$ is continuous and strictly increasing then it's bijection. It's has a reciprocal function g^{-1} called "inverse cosine hyberbolic" is continuous and strictly increasing on $[1, +\infty[$. We write $g^{-1} : x \to arg\cosh(x)$ and we have

$$x \geq 1,\ y = arg\cosh(x) \iff y \geq 0,\ x = \cosh(y)\,.$$

Remark 19. $\forall x \in [1, +\infty[\,,\ arg\cosh x = \ln\left(x + \sqrt{x^2 - 1}\right)$ (logarithm expression). In fact, we have

$$\cosh^2(y) - \sinh^2(y) = 1,$$

and

$$x \in [1, +\infty[\,,\ y = arg\cosh(x) \iff y \in [0, +\infty[\,,\ x = \cosh(y)\,,$$

so,

$$\sinh(y) = \sqrt{x^2 - 1},$$

and since $e^y = \cosh(y) + \sinh(y)$, then

$$e^y = x + \sqrt{x^2 - 1}.$$

Thus,

$$y = \ln\left(x + \sqrt{x^2 - 1}\right) \iff argch(x) = \ln\left(x + \sqrt{x^2 - 1}\right),\ x \geq 1.$$

7.2.2.1 Derivative

La function "$arg\cosh$" is differentiable on $]1, +\infty[$ and it's derivative is defined as

$$f'(x) = \frac{1}{\sqrt{x^2 - 1}}.$$

7.2.2.2 Variation table

x	1		$+\infty$
Argch'x	$+\infty$	+	0^+
Argch x	0	$\nearrow$	$+\infty$

7.2.2.3 Graph

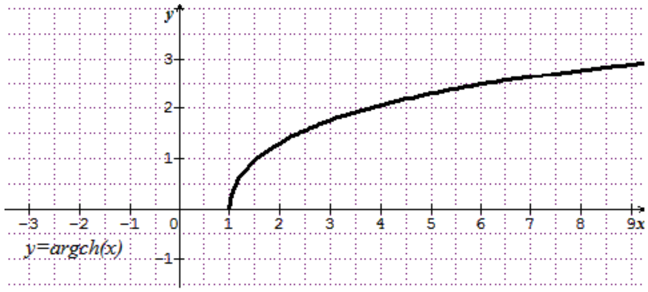


Figure 7.2.2: Graphe argcosh

7.2.3 Inverse hyperbolic tangent function

Definition 36. The function $h : \mathbb{R} \to \left]-1,1\right[$ defined by $h(x) = \tanh(x)$ is continuous and strictly increasing then it's bijection. It has a reciprocal function h^{-1} called "inverse tangent hyberbolic" is continuous and strictly increasing from $\left]-1,1\right[$ into $\mathbb{R}$. We write $h^{-1} : x \mapsto arg\tanh(x)$ and we have

$$x \in \left]-1,1\right[,\ y = arg\tanh(x) \Longleftrightarrow y \in \mathbb{R},\ x = \tanh(y).$$

Remark 20. 1) The function "$argth$" is odd.

2) $\forall x \in \left]-1,1\right[,\ argth(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ (logarithmic expression). Indeed, we have

$$x \in \left]-1,1\right[,\ y = arg\tanh(x) \Longleftrightarrow y \in \mathbb{R},\ x = \tanh(y),$$

and

$$x = \tanh(y) = \frac{\sinh(y)}{\cosh(y)} = \frac{e^{2y}-1}{e^{2y}+1},$$

Then

$$e^{2y} = \frac{1+x}{1-x}, \quad |x| < 1$$

Thus,

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \Longleftrightarrow arg\tanh(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ |x| < 1.$$

7.2.3.1 Derivative

The function "$arg\tanh$" is differentiable on $D = \left]-1,1\right[$ and it's derivative is

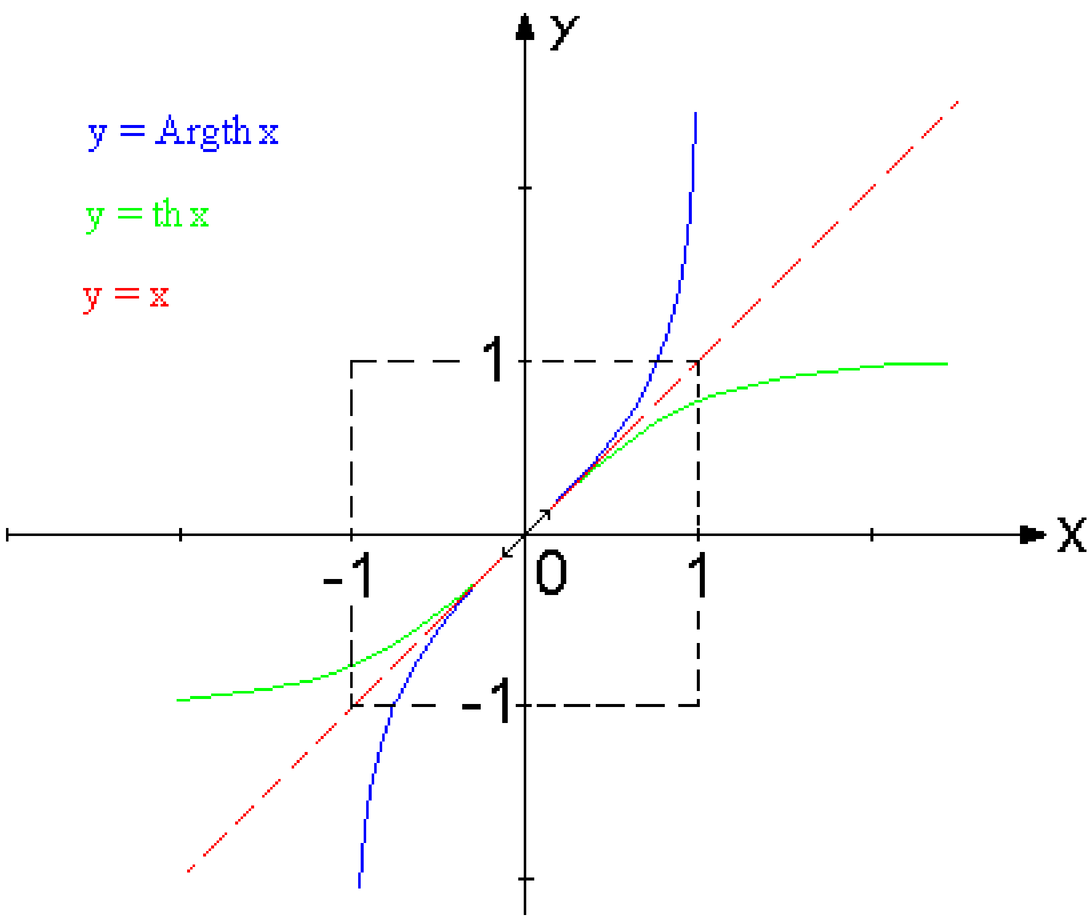
$$f'(x) = \frac{1}{1-x^2}.$$

7.2.3.2 Table of variation

Since the function is odd function, then we can study on $]0, 1[$.

x	0		1
$\text{Argth}'x$	1	$+$	$+\infty$
$\text{Argth}\,x$	0	$\nearrow$	$+\infty$

7.2.3.3 Graph



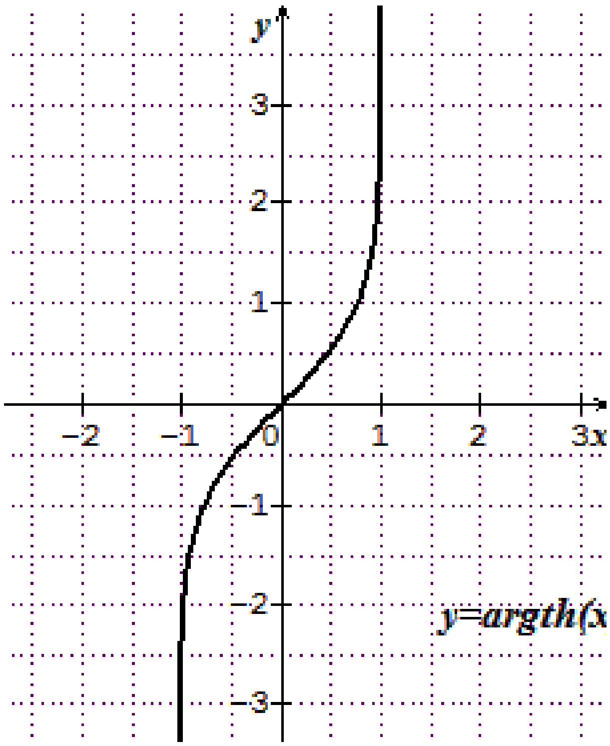


Figure 7.2.3: La courbe représentative de la fonction th et son réciproque argth

7.2.4 Inverse cotangent hyperbplic function

Definition 37. The function $k : \mathbb{R}^* \to \left]-\infty,-1\right[\cup \left]1,+\infty\right[$ defined by $k(x) = coth(x)$ is continuous and strictly increasing then it's bijection. It has a reciprocal function k^{-1} called "inverse cotengent hyperbolic" is continuous and strictly increasing from $\left]-\infty,-1\right[\cup \left]1,+\infty\right[$ into $\mathbb{R}^*$. We write $k^{-1} : x \to argco\tanh(x)$ and we have

$$\forall x \in \left]-\infty,-1\right[\cup \left]1,+\infty\right[,\ y = argco\tanh(x) \Longleftrightarrow \forall y \in \mathbb{R}^*,\ x = co\tanh(x).$$

Remark 21. 1) $\forall x \in \left]-\infty,-1\right[\cup \left]1,+\infty\right[,\ argcoth(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ (logarithmic expression). Indeed, we have

$$x \in \left]-\infty,-1\right[\cup \left]1,+\infty\right[,\ y = argco\tanh(x) \Longleftrightarrow y \in \mathbb{R}^*,\ x = co\tanh(y),$$

and

$$x = \coth(y) = \frac{\cosh(y)}{\sinh(y)} = \frac{e^{2y}+1}{e^{2y}-1},$$

So,

$$e^{2y} = \frac{x+1}{x-1}, \quad |x| < 1.$$

Thus,

$$y = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \Longleftrightarrow arg\tanh(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),\ |x| < 1.$$

7.2.4.1 Derivative

The function "*arg* coth" is differentiable on $D =]-\infty, -1[\cup]1, +\infty[$ and it's derivative defined by

$$f'(x) = \frac{1}{x^2 - 1}.$$

7.2.4.2 Variation table

x	1		$+\infty$
Argcoth'x	$-\infty$	$-$	0^-
Argcoth x	$+\infty$	$\searrow$	0

7.2.4.3 Graph

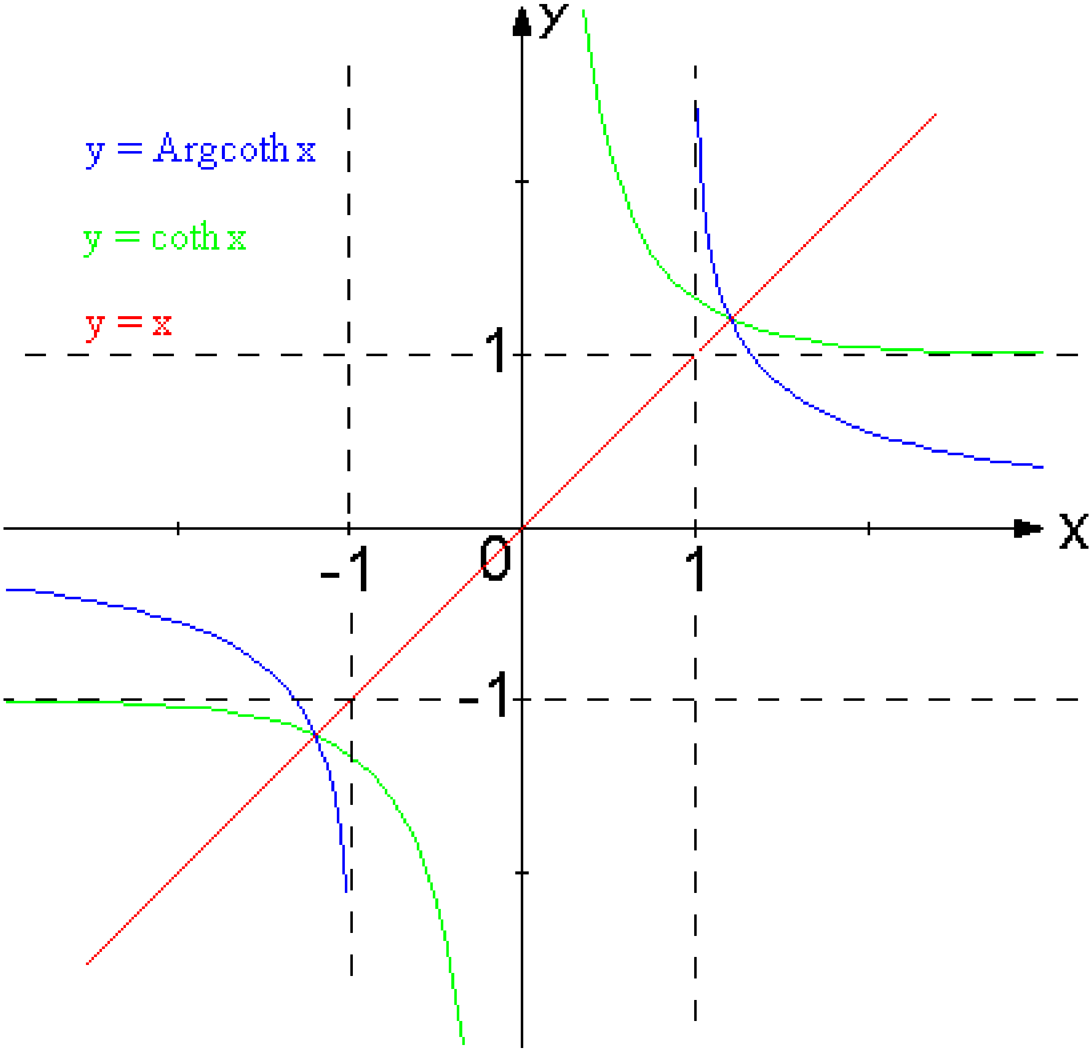


Figure 7.2.4: Graphe argcoth

Exercise 35. Let the function f defined by $f(x) = arg\cosh\left(\sqrt{1+x^2}\right)$.
1) Determine the definition domain of f.
2) Simplify the l'expression.

Solution: 1) Definition domain of f est $D_f = \mathbb{R}$. Car $\forall x \in \mathbb{R},\ \sqrt{1+x^2} > 1$.
2) Simplifcation: We have $\sinh^2(x) = 1 + \cosh^2(x)$. Since "sinh" is bijection, then $\forall x \in \mathbb{R},\quad \exists y \in \mathbb{R}$ such that $x = \sinh(y)$, so,

$$1 + x^2 = 1 + \sinh^2(y) \Longrightarrow \sqrt{1+x^2} = \sqrt{\cosh^2(y)} = \cosh(y).$$

Therefore

$$f(x) = arg\cosh(\cosh(y)) = \begin{cases} y, & if\ y \geq 0, \\ -y, & if\ y < 0. \end{cases}$$

So, since $arg\cosh(x) = \ln\left(x+\sqrt{x^2-1}\right),\ x \geq 1$, then

$$f(x) = arg\cosh(\cosh(y)) = \ln\left(\cosh(y)+\sqrt{\cosh^2(y)-1}\right)$$
$$= \ln\left(\cosh(y)+\sqrt{\sinh^2(y)}\right)$$
$$= \ln(\cosh(y)+|\sinh(y)|),$$

Thus,

$$f(x) = \begin{cases} arg\sinh(x), & if\ x \geq 0, \\ -arg\sinh(x), & if\ x < 0. \end{cases}$$

Exercise 36. Calculate the derivative of the following functions :

1) $f(x) = arg\tanh(x^3+1)$, $f(x) = \arcsin(arg\sinh(x))$, $arg\sinh(x^2+1)$.

7.3 Derivatives of higher orders (Successives derivatives)

Let f be a function n times differentiable on I, $(n \in \mathbb{N})$. Then, the derivative of order n (nth derivative) noted by $f^{(n)}$ is function from I to $\mathbb{R}$. $f^{(n)}$ is also differentiable on I. Then

$$f^{(n+1)} = \left(f^{(n)}\right)'.$$

Remark 22. by convention, we set $f^{(0)} = f$.

Example 31. 1) The function $f(x) = \sin(x)$ is n times differentiable on $\mathbb{R}$ and we have

$$f^{(n)}(x) = \sin\left(x+n\frac{\pi}{2}\right),\ \forall n \in \mathbb{N}.$$

is differentiable on $\mathbb{R}$ and consequently, we have

$$f^{(n+1)}(x) = \left(f^{(n)}\right)' = \cos\left(x+n\frac{\pi}{2}\right) = \sin\left(x+(n+1)\frac{\pi}{2}\right).$$

This last property can be proved by recurrencente.

2) The function $f(x) = \cos(x)$ is n times differentiable on $\mathbb{R}$ and we have

$$f^{(n)}(x) = \cos\left(x+n\frac{\pi}{2}\right),\ \forall n \in \mathbb{N}.$$

Exercise 37. Calculate the derivative of order n for the following functions :
1) $f(x) = \frac{1}{1-x}$,
2) $g(x) = \frac{1}{\sqrt{1-x^2}}$.

solution: the functions f, g are n times differentiable on I, J respectively and we have
1) $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$, $\forall x \in I = \mathbb{R}-\{1\}$, $\forall n \in \mathbb{N}$.
2) $g^{(n)}(x) = \frac{p_n(x)}{(1-x)^{n+1}}$, $\forall x \in J =]-1,1[$, $\forall n \in \mathbb{N}$, où $p_{n+1}(x) = (1-x^2)p_n(x) + n^2(1-x^2)p_{n-1}(x)$.

7.3.1 Leibniz formula

Theorem 18. *Let f, g two functions n times differentiable on $I \subset \mathbb{R}$. Then, $f \cdot g$ is n times differentiable on I. Furthermore, for each naturel number n, we have*

$$\begin{aligned}(f \cdot g)^{(n)} &= \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)} \\ &= f^{(n)} g + C_n^1 f^{(n-1)} g' + C_n^2 f^{(n-2)} g'' + \cdots + C_n^{n-1} f' g^{(n-1)} + f g^{(n)}.\end{aligned}$$

Exercise 38. Soit $y = x \sinh(x)$.
- Calculate $y^{(100)}$.

Solution: From Leibniz formula, we have

$$\begin{aligned}y^{(100)} = \sum_{k=0}^{100} C_{100}^k x^{(k)} (\sinh(x))^{(100-k)} &= x \sinh(x) + C_{100}^1 \cosh(x) \\ &= x \sinh(x) + 100 \cosh(x)\end{aligned}$$

7.4 Functions of a class C^n

Definition 38. Let $I \subset \mathbb{R}$ and $n \in \mathbb{N}^*$. A function $f : I \to \mathbb{R}$ is called of a class C^n on I if it is $n - times$ differentiable and I and $f^{(n)}$ is continuous on I.

Remark 23. 1) For $n = 0$, f of a class C^0 on I if f is a continuous on I.
2) In particulary case when f is infinitly differentiable on I, we say that f is of a class C^∞ on I. For example, $\sin \in C^\infty(\mathbb{R}, \mathbb{R})$, (respectively $\cos \in C^\infty(\mathbb{R}, \mathbb{R})$).

7.5 Taylor-Lagrange Formula

Theorem 19. *Let $[a, b]$ be an closed interval $]a, b[$ be an open interval. Suppose that $f \in C^n([a, b])$ and $f^{(n)}$ is differentiable on $]a, b[$. There exists $c \in]a, b[$ such that*

$$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \cdots + \frac{(b-a)^n}{n!} f^{(n)}(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

is called nth order Taylor polynomial with Lagrange form of the remainder

$$\frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(c),$$

is called the remainder term in the formula.

Remark 24. 1) We note that Taylor polynomial is a good approximation to f.

2) Taylor's theorem gives us a quick proof of a version of the second derivative test. By a strict relative minimum of f at a, we mean that there exists a $\delta > 0$ such that $f(x) > f(a)$ for all $x \in]a - \delta, a + \delta[$ here $x \neq a$. A strict relative maximum is defined similarly. Continuity of the second derivative is not needed.

Proposition 22. *(Second derivative test(Another sufficient condition)). Suppose* $f :]a,b[\to \mathbb{R}$ *is twice continuously differentiable,* $x_0 \in]a,b[$ *and* $f'(x_0) = 0$.

(i) If $f''(x_0) > 0$*. Then* f *has a strict relative minimum at* x_0.

(ii) If $f''(x_0) < 0$*. Then* f *has a strict relative maximum (local maximum) at* x_0.

Remark 25. (ii)The conditions given in Proposition 23 are only sufficient conditions. There are functions f for which none of the conditions (i) and (ii) are satisfied at a point x_0, still f can have local extremum at x_0. For example, consider

$$f(x) = x^4, \quad \text{and } g(x) = 1 - x^4, \quad |x| < 1.$$

Then $f'(0) = 0 = g'(0)$, f has local minimum at 0 and g has local maximum at 0. But $f''(0) = 0 = g''(0)$

Theorem 20. *There are other ways to write the remainder term, but we skip those. Note that* c *depends on both* x *and* a.

7.6 Taylor-Mac-Laurin Formula

1) In Taylor Formula, we put $h = b - a,\ \text{and } c = a + \theta h,\ \theta \in]0,1[$, we obtain the following formula

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta h).$$

2) If in the above formula, we interchange a by 0, we have $\forall h \in [a,b]$, there is $\theta \in]0,1[$ such that

$$f(h) = f(0) + \frac{h}{1!}f'(0) + \cdots + \frac{h^n}{n!}f^{(n)}(0) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(\theta h),$$

this relation called Mac-Laurin formula.

7.7 Taylor-Yong Formula

Theorem 21. *Let* $I \supset \mathbb{R}$ *and* $a \in I$*. Suppose that* f *of a class* C^n *on* I *and* f *is differentiable* $(n+1) - $*times. Then* $\forall (a+h) \in I$*, there is a reel function* ϵ *such that*

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \frac{h^{n+1}}{(n+1)!}\left(f^{(n+1)}(a) + \epsilon(h)\right),$$

with $\lim\limits_{h \to 0} \epsilon(h) = 0$.
is called Taylor-Yong Formula of order n *with Yong reminder*

$$\frac{h^{n+1}}{(n+1)!}\left(f^{(n+1)}(a) + \epsilon(h)\right).$$

Example 32. Let $f(x) = \sin(x)$. We know that f is of a class C^∞. $\forall k \in \mathbb{N}$, we have

$$f^{(k)}(x) = \sin\left(x + k\frac{\pi}{2}\right).$$

Consequently, for $x = 0$, we have

$$f^{(k)}(0) = \sin\left(k\frac{\pi}{2}\right).$$

If $k = 2p$, then $f^{(2p)}(0) = \sin(0) = 0$.
if $k = 2p+1$, then $f^{(2p+1)}(0) = \sin\left(\frac{\pi}{2} + p\pi\right) = (-1)^p$.
Now, for $x \in \mathbb{R}$

$$f^{(2p+2)}(x) = \sin(x + (p+1)\pi) = (-1)^{p+1}\sin(x).$$

Consequently, the Mac-Laurin formula with Lagrange reminder is, $\forall x \in \mathbb{R}$, $\exists \theta \in]0,1[$, such that

$$\sin(x) = x - \frac{x^3}{3!} + \cdots + (-1)^p \frac{x^{2p+1}}{(2p+1)!} + (-1)^{p+1} \frac{x^{2p+2}}{(2p+2)!}\sin(\theta x).$$

2) Similarly, we have $\forall x \in \mathbb{R}$, $\exists \theta \in]0,1[$, such that

$$\cos(x) = 1 - \frac{x^2}{2!} + \cdots + (-1)^p \frac{x^{2p}}{(2p)!} + (-1)^{p+1} \frac{x^{2p+1}}{(2p+1)!}\sin(\theta x).$$

is called Mac-Laurin formula with Lagrange remainder for the function $f : x \mapsto \cos(x)$.

Chapter 8

Approximations by polynomials (Taylor polynomials)

Polynomials are generally a good choice for an approximating function since they are so easy to work with. Depending on the situation other families of functions may be more appropriate. For example if you are approximating a periodic function, then sums of sines and cosines might be a better choice; this leads to Fourier series.
Any polynomial in x of degree n can also be expressed as a polynomial in $(x-a)$ of the same degree n and vice versa. So $P_n(x)$ really still is a polynomial of degree n.

We can use the same strategy to generate still better approximations by polynomials of any degree we like. We determine the coefficients of the polynomial by requiring, that at the point $x=a$, the approximation and its first n derivatives agree with those of the original function.

Definition 39. : Let f be a numerical function defined near $x=0$, except perhaps in 0. It is said that f admits a limited expansion of order n near 0 if there exists a polynomial of degree not more than n and a function ϵ such that for any element x near 0, we have

$$f(x) = p(x) + x^n \varepsilon(x)\,,\ and\ \lim_{x\to 0} \varepsilon(x) = 0,$$

where $p(x) = a_0 + a_1 x + a_2 x + \cdots + a_n x^n,\ a_i \in \mathbb{R}$, $(i = 0, 1, \ldots, n)$, is the polynomial function $x \mapsto p(x)$.

Proposition 23. *(unicity) If f admit a limited expansion of order n near 0, then it's unique.*

Remark 26. 1) We can write $o(x^n)$ instead of $x^n\varepsilon(x)$ and we have

$$f(x) = a_0 + a_1 x + a_2 x + \cdots + a_n x^n + o(x^n)\,,\ and \lim_{x\to 0} \frac{o(x^n)}{x^n} = 0.$$

2) We can define a limited expansion of order n for the function f near $x = x_0 \in \mathbb{R}$ via the change of variable $t = x - x_0$for $x \to x_0$we have $t \to 0$. We have the following definition.

Definition 40. Let f be real valued function near $x = x_0 \in I \subset \mathbb{R}$, except perhaps in x_0. We say that f admits a limited expansion of order n near x_0 if there is a real numbers $b_0, b_1, \ldots, b_n$ and an function ε such that for any element x near x_0, we have

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \cdots + b_n(x - x_0)^n + (x - x_0)^n \varepsilon(x - x_0), \textit{and} \lim_{x \to x_0} \varepsilon(x - x_0) = 0.$$

8.1 Usual limited expansion obtained by the formula Mac-Laurin-Yong

Theorem 22. *Let I an open interval of $\mathbb{R}$ contenant 0 and f is a function defined on I into $\mathbb{R}$ of a class C^{n-1} in I and has a derivative of order n in $x = 0$. Then, f admit a limited expansion of order n near 0 is*

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{x^n}{n!}\varepsilon(x) \ \ \textit{avec} \ \lim_{x \to 0} \varepsilon(x) = 0.$$

With Landau notation, we have

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + o(x^n).$$

Example 33. 1) The function $x \longmapsto e^x$ is infinitely differentiable on $\mathbb{R}$ and they derivative of order n tends to 1 at point 0, we have the limited expansion of n near 0, is

$$e^x = 1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!} + o(x^n).$$

2) The function $x \longmapsto \sin(x)$ is infinitely differentiable on $\mathbb{R}$ and they derivative of order n est $\sin\left(x + n\frac{\pi}{2}\right)$, we have the limited expansion of $n = 2p + 1$ near 0,

$$\sin(x) = x - \frac{x^3}{3!} + \cdots + (-1)^p \frac{x^{2p+1}}{(2p+1)!} + o\left(x^{2p+1}\right).$$

3) The function $x \longmapsto \cos(x)$ is infinitely differentiable on $\mathbb{R}$ and they derivative of order n est $\cos\left(x + n\frac{\pi}{2}\right)$, we have the limited expansion of $n = 2p$ near 0,

$$\cos(x) = 1 - \frac{x^2}{2!} + \cdots + (-1)^p \frac{x^{2p}}{(2p)!} + o\left(x^{2p}\right).$$

4) The function $x \longmapsto (1 + x)^\alpha \in C^\infty(]-1, +\infty[, \mathbb{R})$, $\alpha \in \mathbb{R}$ and they derivative of order n is

$$f^{(n)}(x) = \alpha(\alpha - 1)\cdots(\alpha - n + 1)(1 + x)^{\alpha - n},$$

we have the limited expansion of n near 0,

$$(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha - 1)x^2}{2!} + \cdots + \frac{\alpha(\alpha - 1)\cdots(\alpha - p + 1)x^n}{n!} + o(x^n).$$

If $\alpha = \frac{1}{2}$ and $\alpha = -\frac{1}{2}$. We have the limited expansion of n near 0

$$(1 + x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1 \times 1}{2 \times 4}x^2 + \cdots + (-1)^{n-1}\frac{1 \times 1 \times 3 \times \cdots \times (2n - 3)}{2 \times 4 \times 6 \times \cdots \times 2n}x^n + o(x^n),$$

and

$$(1 + x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{1 \times 3}{2 \times 4}x^2 + \cdots + (-1)^n \frac{1 \times 1 \times 3 \times \cdots \times (2n - 1)}{2 \times 4 \times 6 \times \cdots \times 2n}x^n + o(x^n).$$

8.2 Integration of limited expansion

Theorem 23. *Suppose that the function f admit we have the limited expansion whose part principal is p of order n near 0 and, moreover, f is integrable on a closed interval containing 0. Then $\int_0^x f(t)\,dt$ admit near 0,, a limited expansion of order $n+1$ whose main part is $\int_0^x p(t)\,dt$.*

8.2.1 Applications

1) We will determine the limited expansion of $x \longmapsto \ln(1+x)$ and $x \longmapsto arg\tanh(x)$.
By dividing according to the increasing powers of order n of 1 by $1-x$, we get

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \frac{x^{n+1}}{1-x}.$$

Consequently, in interval I containing 0 and not containing 1, for $x \in I$, we get

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + o(x^n).$$

Thanks to the Theorem 20, we have

$$\int_0^x \frac{dt}{1-t} = \int_0^x \left(1 + t + t^2 + t^3 + \cdots + t^n\right) + o\left(x^{n+1}\right),$$

Let

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n+1} + o\left(x^{n+1}\right). \tag{8.1}$$

By changing x by $-x$, we obtain in any interval I' containing 0 and not contain -1, for $x \in I'$, the following relation,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^n \frac{x^n}{n+1} + o\left(x^{n+1}\right). \tag{8.2}$$

By addition (8.1) and (8.2), we obtain in any interval J contain 0 and $1 \notin J$, $-1 \notin J$, for $x \in J$, the following relation,

$$arg\tanh(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \cdots + \frac{x^{2n+1}}{2n+1} + o\left(x^{2n+2}\right).$$

2) We will determine the limited expansion of $x \longmapsto \arctan(x)$.
For all $x \in \mathbb{R}$, we have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \cdots + (-1)^n x^{2n} + o\left(x^{2n+1}\right).$$

Consequently, thanks to the Theoreme 20, we have

$$\arctan(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{2n}}{2n+1} + o\left(x^{2n+2}\right).$$

3) We will determine the limited expanssion of $x \longmapsto \arcsin(x)$.
In Section 8.1, we see that for $x \in J$ (J an interval satisfied $1 \notin J$, $-1 \notin J$ and $0 \in J$),

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1 \times 3}{2 \times 4}x^4 + \cdots + \frac{1 \times 1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times 2n}x^{2n} + o\left(x^{2n+1}\right).$$

Integrating, for $x \in J$, we have

$$\arcsin(x) = x + \frac{1}{2 \times 3}x^3 + \frac{1 \times 3}{2 \times 4 \times 5}x^5 + \cdots + \frac{1 \times 1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times 2n \times (2n+1)}x^{2n+1} + o\left(x^{2n+2}\right).$$

Remark 27. By integration, the first member write also

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} = [-\arccos(t)]_0^x = -\arccos(x) + \frac{\pi}{2}.$$

Consequently,

$$\arccos(x) = \frac{\pi}{2} - \left[x + \frac{1}{2 \times 3}x^3 + \frac{1 \times 3}{2 \times 4 \times 5}x^5 + \cdots + \frac{1 \times 1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times 2n \times (2n+1)}x^{2n+1}\right] + o\left(x^{2n+2}\right).$$

4) We will determine the limited expansion of $x \longmapsto arg\sinh(x)$. By similar maner, we obtain

$$\frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2}x^2 + \frac{1 \times 3}{2 \times 4}x^4 - \cdots + (-1)^n \frac{1 \times 1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times 2n}x^{2n} + o\left(x^{2n+1}\right).$$

Integrating, we have

$$arg\sinh(x) = x - \frac{1}{2 \times 3}x^3 + \frac{1 \times 3}{2 \times 4 \times 5}x^5 - \cdots + (-1)^n \frac{1 \times 1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times 2n \times (2n+1)}x^{2n+1} + o\left(x^{2n+2}\right).$$

8.3 Derivation of limited expansion

Theorem 24. *Suppose that the function f admit we have the limited expansion whose part principal is p of order n near 0 and, moreover, f is integrable on a closed interval containing 0. Then f' admit near 0, a limited development $n-1$ whose main part is p'.*

8.3.1 Applications

1) We will determine the limited expansion of $x \longmapsto \frac{1}{(1-x)^2}$ and of $x \longmapsto arg\tanh(x)$.
According to the limited expansion of order n of the function $x \longmapsto \frac{1}{1-x}$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + o(x^n).$$

By differentiating (use the Theoreme 21), we have

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + o\left(x^{n-1}\right).$$

8.4 Operations of limited expansion

8.4.1 Addition

Proposition 24. *If near $x = 0$, f admit a limited expansion whose part principal is p of order n and g admit a limited expansion whose part principal is q of order n. Then, the function $f + g$ admit near $x = 0$, a limited expansion whose part principal is $p + q$ of order n. Indeed,*

$$f(x) = p(x) + o(x^n), \quad \textit{and} \quad g(x) = q(x) + o(x^n),$$

by addition, we obtain

$$(f + g)(x) = p(x) + q(x) + o(x^n),$$

since $p + q$ is a polynomial of degree less than or equal to n.

Example 34. We know that

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

On the other hand,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + o(x^n),$$

and

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^n}{n!} + o(x^n).$$

By addition firstly and soustraction secondly, we obtain

$$\cosh(x) = 1 + \frac{x^2}{2!} + \cdots + \frac{x^{2p}}{(2p)!} + o\left(x^{2p+1}\right)$$

$$\sinh(x) = 1 + \frac{x^3}{3!} + \cdots + \frac{x^{2p+1}}{(2p+1)!} + o\left(x^{2p+2}\right).$$

8.4.2 Product

Proposition 25. *If near $x = 0$, f admit a limited expansion whose part principal is p of order n and g admit a limited expansion whose part principal is q of order n. Then, the function fg admit near $x = 0$, a limited expansion whose part principal is pq of order by removing from the compound polynomial pq the terms of degree n ((remainder of the Euclidean division pq par x^{n+1}). In fact,*

$$f(x) = p(x) + o(x^n), \qquad \textit{and} \qquad g(x) = q(x) + o(x^n), \tag{8.3}$$

Since, the polynomial functions $x \mapsto p(x)$ and $x \mapsto q(x)$ are bounded near 0, we have

$$p \cdot o(x^n) = o(x^n), \; q \cdot o(x^n) = o(x^n), \quad o(x^n) \cdot o(x^n) = o(x^n).$$

So, by multiplication the relations (8.3), we obtain

$$(fg)(x) = p(x)\,q(x) + o(x^n).$$

Let $r(x)$ the remainder of the euclidean division pq by x^{n+1} and by removing from the product polynomial pq the terms of degree n, we obtain

$$p(x)\,q(x) = r(x) + o(x^n).$$

Therefore,

$$f(x)\,g(x) = r(x) + o(x^n).$$

Example 35. We will determine the limited expansion near 0 of order 3 for the function

$$f(x) = \frac{e^x}{\sqrt{1+x}}.$$

We have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + o\left(x^3\right),$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{8}x^2 - \frac{5}{16}x^3 + o\left(x^3\right).$$

According tothe proposition 22, we obtain by multiplication

$$\frac{e^x}{\sqrt{1+x}} = 1 + \left(1 - \frac{1}{2}\right)x + \left(\frac{1}{2} - \frac{1}{2} + \frac{3}{8}\right)x^2 - \left(\frac{1}{6} - \frac{1}{4} + \frac{3}{8} - \frac{5}{16}\right)x^3 + o\left(x^3\right)$$

$$\frac{e^x}{\sqrt{1+x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 - \frac{1}{48}x^3 + o\left(x^3\right).$$

8.4.3 Quotient

Proposition 26. *If, near $x = 0$, f admit a limited expansion whose part principal is p of order n and g admit a limited expansion whose part principal is q of order n, then the function $\frac{f}{g}$ (if $q(0) \neq 0$) admit near $x = 0$, a limited expansion of order n obtained, by the division according to increasing powers of the polynomial p by the polynomial q.*

Example 36. We will determine near $x = 0$ the limited expansion, of order 5, of the function $x \longmapsto \tan(x)$. We have

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + o\left(x^5\right),$$

and

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o\left(x^5\right).$$

By the division to the order 5, according to increasing powers, we obtain

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + o\left(x^5\right).$$

Similarly, we write the limited expansion of order 5, of the function $x \longmapsto \sinh(x)$ and $x \longmapsto \cosh(x)$, we obtain the limited expansion of order 5, of the function $x \longmapsto \tanh(x)$,

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} + o\left(x^5\right).$$

8.5 Composition of limited expansion

Proposition 27. *If near $x = 0$, f admit a limited expansion whose part principal is p of order n and g admit a limited expansion whose part principal is q of order n and $\lim\limits_{x\to 0} f(x) = 0$. Then, the function $g \circ f$ admit near $x = 0$, a limited expansion of order n obtained, by removing from the composite polynomial $q \circ p$ the terms greater than to n (remainder of the Euclidean division of $q \circ p$ by x^{n+1}).*

Example 37. We will determine near $x = 0$ the limited expansion, of order 5 of the function $x \longmapsto \sinh[\ln(1+x)]$.
On a

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + o\left(x^5\right),$$

$$\sinh(u) = u - \frac{u^3}{6} + \frac{u^5}{120} + o\left(u^5\right),$$

we replace u by the limited expansion of $\ln(1+x)$ and using the proposition 25, we obtain

$$\sinh[\ln(1+x)] = x - \frac{x^2}{2} + \frac{x^3}{2} - \frac{x^4}{2} + \frac{x^5}{2} + o\left(x^5\right).$$

Remark 28. We can obtaind the last limited expansion by using the follwing relation:

$$\sinh[\ln(1+x)] = \frac{1}{2}\left(e^{\ln(1+x)} - e^{-\ln(1+x)}\right) = \frac{1}{2}\left(1 + x - \frac{1}{1+x}\right).$$

Since

$$-\frac{1}{1+x} = -1 + x - x^2 + x^3 - x^4 + x^5 + o\left(x^5\right),$$

we have

$$\sinh[\ln(1+x)] = x - \frac{x^2}{2} + \frac{x^3}{2} - \frac{x^4}{2} + \frac{x^5}{2} + o\left(x^5\right).$$

Exercise 39. Using the limited expansion of usual s functions usuelles, determine the limited expansion near 0 of order 4, for the follwing functions:

1) $f(x) = \frac{1+x+x^2}{1-x+x^2}$,
2) $g(x) = \ln\left(\frac{\sin(x)}{x}\right)$,
3) $h(x) = \ln\left(\frac{sh(x)}{x}\right)$,
4) $k(x) = \frac{x}{e^x-1}$,
5) $l(x) = (1+x)^{\frac{1}{x}}$.

Solution: 1) There is many methods to obtain the limited expansion of the function f.

a) First method: By the division according the increasing powers x, of $1 + x + x^2$ by $1 - x + x^2$, we obtain

$$f(x) = 1 + 2x + 2x^2 - 2x^4 + o\left(x^4\right).$$

b) Second method: We use the limited expansion near 0, of the function $u \mapsto (1+u)^{-1}$ with $u = -x + x^2$ and the limited expansion obtaind by the product of two functions, we obtain

$$f(x) = \left(1 + x + x^2\right)\left(1 + \left[-x + x^2\right]\right)^{-1}$$

$$f(x) = \left(1 + x + x^2\right)\left(1 + \left[-x + x^2\right] + \left(-x + x^2\right)^2 - \left(-x + x^2\right)^3 + \left(-x + x^2\right)^4 + o\left(\left(-x + x^2\right)^4\right)\right),$$

by removing from the obtained product polynomial the tems of degree greater than 4, we obtain

$$f(x) = \left(1 + x + x^2\right)\left(1 + x - x^3 - x^4 + o\left(x^4\right)\right),$$

also, by removing from the product polynomial the tems of degree greater than 4, we obtain

$$f(x) = 1 + 2x + 2x^2 - 2x^4 + o\left(x^4\right).$$

2) First, we write the limited expansion near 0, of order 5, of the function $x \longmapsto \sin(x)$, then dividing over x, we obtain

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + o\left(x^4\right).$$

So, since

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} + o\left(u^3\right),$$

then replaces u with $-\frac{x^2}{6} + \frac{x^4}{120} + o\left(x^4\right)$ and using the limited expansion theorem of a composite function, it follows that

$$\ln\left(\frac{\sin(x)}{x}\right) = -\frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^4}{2(3!)^2} + o\left(x^4\right),$$

so,

$$\ln\left(\frac{\sin(x)}{x}\right) = -\frac{x^2}{6} + \frac{x^4}{180} + o\left(x^4\right).$$

3) Similarly to 2).
4) We use the limited expansion near 0 of order 5, for the function $x \longmapsto e^x$, we obtain

$$e^x - 1 = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + o\left(x^5\right).$$

We use the division according to increasing powers of x, of the main part x with $\frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$, we obtain

$$k(x) = \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + o\left(x^4\right).$$

5) We write the function in the following formula

$$l(x) = e^{\frac{1}{x}\ln(1+x)}$$

Since

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + o\left(x^5\right),$$

according to the theorem of limited expansion of quotient of two functions, we obtain

$$\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} + o\left(x^4\right),$$

the limited expansion of the function $u \longmapsto e^u$ near 0, and remplaces u with $\left(-\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} + o\left(x^4\right)\right)$, and using the limited expansion theorem of a composite function, it follows that (by removing from the composite polynomial the tems of degree greater than 4)

$$(1+x)^{\frac{1}{x}} = e^{\frac{\ln(1+x)}{x}} = ee^{-u} = e - \frac{ex}{2} + \frac{11ex^2}{24} - \frac{7ex^3}{16} + \frac{2447ex^4}{5760} + o\left(x^4\right),$$

Exercise 40. Calculate the limited expansion of order 4, near 0 for the following functions :

1) $f(x) = \frac{e^{\arctan(x)}}{1+x^2}$
2) $g(x) = e^{\cos(x)}$
3) $h(x) = \sqrt[3]{1+x+x^2}$
4) $k(x) = \ln(\cos(x))$
5) $l(x) = \arctan(\sin(x))$, to the order 5.

Solution: 1) $f(x) = \frac{e^{\arctan(x)}}{1+x^2}$. The function $x \longmapsto e^{\arctan(x)}$ is composition of two functions, we have

$$e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + o\left(u^4\right),$$

and since

$$\arctan(x) = x - \frac{x^3}{3} + o\left(x^4\right),$$

then,

$$e^{\arctan(x)} = e^{x - \frac{x^3}{3} + o\left(x^4\right)},$$

and we replace u with $x - \frac{x^3}{3} + o(x^4)$ and according to the compositional theorem of two functions for limited expansion, it follows that

$$e^{\arctan(x)} = 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{7}{24}x^4 + o\left(x^4\right).$$

We use the division according to increasing powers of x of the main part $1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{7}{24}x^4 + o(x^4)$ by $1 + x^2$, we obtain
2) $g(x) = e^{\cos(x)}$. The function g is composition of two functions, we have

$$e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + o\left(u^5\right),$$

and since

$$\cos(x) = 1 - \frac{x^2}{4} + \frac{x^4}{24} + o\left(x^4\right),$$

then,

$$e^{\cos(x)} = e\left(e^{-\frac{x^2}{4} + \frac{x^4}{24} + o\left(x^5\right)}\right),$$

and we replace u with $-\frac{x^2}{6}+\frac{x^4}{24}+o(x^4)$ and according to the compositional theorem of two functions for limited expansion, it follows that

$$e^{\cos(x)} = e - \frac{e}{4}x^2 + \frac{e}{6}x^4 + o\left(x^4\right),$$

5) $l(x) = \arctan(\sin(x))$. The function l is composition of two functions, we have

$$\arctan(u) = u - \frac{u^3}{3} + \frac{u^5}{5} + o\left(u^5\right),$$

and $u = \sin(x)$ such that

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + o\left(x^5\right).$$

D'où, en replace u par $x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$ and according to the compositional theorem of two functions for limited expansion, it follows that

$$l(x) = \arctan(u) = x - \frac{x^3}{2} + \frac{3x^5}{8} + o\left(x^5\right).$$

Exercise 41. Calculate the limited expansion to the order 3 of thevfollowing functions :

1) $f(x) = \sqrt{\tan(x)}$, near $x_0 = \frac{\pi}{4}$.
2) $g(x) = \arccos(x)$, near $x_0 = 1$.
2) $h(x) = \frac{x^2-1}{x(x+2)}$, near $x_0 = +\infty$.

Solution: 1) By making the change of variable $h = x - \frac{\pi}{4} \to 0,\ as\ x \to \frac{\pi}{4}$. We have

$$\begin{aligned} f(x) = \sqrt{\tan(x)} &= \sqrt{\tan\left(h + \frac{\pi}{4}\right)} \\ &= \sqrt{\frac{1+\tan(h)}{1-\tan(h)}} \\ &= \sqrt{\frac{1+h+\frac{h^3}{3}+o(h^3)}{1-h-\frac{h^3}{3}+o(h^3)}} \\ &= \sqrt{1+2h+2h^2+\frac{8}{3}h^3+o(h^3)} \\ &= 1+h+\frac{1}{2}h^2+\frac{5}{6}h^3+o\left(h^3\right). \end{aligned}$$

Therefore

$$f(x) = \sqrt{\tan(x)} = 1 + \left(x-\frac{\pi}{4}\right) + \frac{1}{2}\left(x-\frac{\pi}{4}\right)^2 + \frac{5}{6}\left(x-\frac{\pi}{4}\right)^3 + o\left(\left(x-\frac{\pi}{4}\right)^3\right)$$

8.6 Generalized limited expansion

Let the function $f : x \longmapsto f(x)$ defined near 0 except at 0. Suppose that $\exists r \in \mathbb{R}^*$ such that the function $x \longmapsto x^r f(x)$ admit (has) a limited expansion of order $n > r$ near 0,

$$x^r f(x) = a_0 + a_1 x + a_2 x + \cdots + a_n x^n + o(x^n).$$

Then, we have

$$f(x) = \frac{1}{x^r}(a_0 + a_1 x + a_2 x + \cdots + a_n x^n) + o\left(x^{n-r}\right).$$

This means that we obtained a generalized limited expansion of order $n - r$ of the function $f : x \longmapsto f(x)$ near 0.

Example 38. The generalized limited expansion of $x \longmapsto \cot(x)$ near $x = 0$, we

$$\cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)}{x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^6)}.$$

If $r = 1$, it follows that

$$x\cot(x) = \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)}{1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^5)} = 1 - \frac{x^2}{3} - \frac{x^4}{45} + o\left(x^5\right).$$

Therefore, we have the generalized limited expansion of the function $x \longmapsto \cot(x)$ near $x = 0$ to the order 4, is

$$\cot(x) = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} + o\left(x^4\right).$$

Exercise 42. 1) Calculate the generalized limited expansion of order 3 near 0, for the follwing functions :

a) $f(x) = \frac{\cos(x)}{\ln(1+x)}$

b) $f(x) = \frac{1}{\ln(1+\tan(x))}$.

2) Calculate the generalized limited expansion of order 3 near infinity, for the following functions :

a) $f(x) = \arctan\left(\sqrt{\frac{x+1}{x+2}}\right)$.

b) $f(x) = \frac{x^3-2}{x+1}$.

Solution: 1) a) $f(x) = \frac{\cos(x)}{\ln(1+x)}$. We see that vnear 0, we have $x\cos(x) \sim x$ et $ln(1+x) \sim x$. We calculate the limited expansion of $f(x) = \frac{x\cos(x)}{\ln(1+x)}$ of order 5, to obtained the limited expansion of $f(x) = \frac{x\cos(x)}{\ln(1+x)}$ to the order 3, we will divide with x two times succesively. We have

$$x\cos(x) = x - \frac{x^3}{2} - \frac{x^5}{24} + o\left(x^5\right),$$

et

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + o\left(x^5\right).$$

After division, we obtain

$$\frac{x\cos(x)}{\ln(1+x)} = 1 + \frac{x}{2} - \frac{7x^2}{12} - \frac{5x^3}{24} - \frac{41x^4}{720} + o\left(x^4\right).$$

So,

$$\frac{\cos(x)}{\ln(1+x)} = \frac{1}{x} + \frac{1}{2} - \frac{7x}{12} - \frac{5x^2}{24} - \frac{41x^3}{720} + o\left(x^3\right).$$

b) $f(x) = \frac{1}{\ln(1+\tan(x))}$. We see that near 0, we have $\ln(1+\tan(x)) \sim \sin(x)$. We calculate the limited expansion $f(x) = \frac{x}{\ln(1+\tan(x))}$ to the order 4, to obtained a limited expansion of the function $x \mapsto \ln(1+\tan(x))$ to the order 5 because we divide with x two times succesively. If we set $u = \tan(x)$, then

$$u = x + \frac{x^3}{3} + \frac{2x^5}{15} + o\left(x^5\right).$$

Since

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \frac{u^5}{5} + o\left(u^5\right).$$

Then

$$\ln(1+\tan(x)) = x - \frac{x^2}{2} + \frac{2x^3}{3} - \frac{7x^4}{12} + \frac{2x^5}{3} + o\left(x^5\right).$$

By making the division according to the increasing powers, we have

$$\frac{x}{\ln(1+\tan(x))} = 1 + \frac{x}{2} - \frac{5x^2}{3} + \frac{x^3}{24} + \frac{11x^4}{144} + o\left(x^4\right),$$

so,

$$\frac{1}{\ln(1+\tan(x))} = \frac{1}{x} + \frac{1}{2} - \frac{5x}{12} + \frac{x^2}{24} + \frac{11x^3}{144} + o\left(x^3\right).$$

2) a) $f(x) = \arctan\left(\sqrt{\frac{x+1}{x+2}}\right)$. If we set $X = \frac{1}{x}$, then $f(X) = \arctan\left(\sqrt{\frac{1+X}{1+2X}}\right)$. We wiil determine the limited expansion of $f(X)$ to the order 3 near 0. By making the division according to the increasing powers, we have

$$\frac{1+X}{1+2X} = 1 - X + 2X^2 - 4X^3 + o\left(X^3\right).$$

We set (put) $u = -X + 2X^2 - 4X^3 + o(X^3)$ tends 0 as X tends 0. Since

$$(1+u)^{\frac{1}{2}} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + \frac{1}{16}u^3 + o\left(u^3\right),$$

then, we have

$$\sqrt{\frac{1+X}{1+2X}} = 1 - \frac{X}{2} + \frac{7X^2}{8} - \frac{25X^3}{16} + o\left(X^3\right).$$

Therefore

$$\arctan\left(\sqrt{\frac{1+X}{1+2X}}\right) = \arctan\left(1 - \frac{X}{2} + \frac{7X^2}{8} - \frac{25X^3}{16} + o\left(X^3\right)\right).$$

If we set $v=-\frac{X}{2}+\frac{7X^2}{8}-\frac{25X^3}{16}+o\left(X^3\right)$ tends to 0 as X tends to 0. Since

$$\arctan\left(1+v\right)=\frac{\pi}{4}+\frac{v}{2}-\frac{v^2}{4}-\frac{v^3}{12}+o\left(v^3\right),$$

then, we have

$$\arctan\left(\sqrt{\frac{1+X}{1+2X}}\right)=\arctan\left(1-\frac{X}{2}+\frac{7X^2}{8}-\frac{25X^3}{16}+o\left(X^3\right)\right)$$
$$=\frac{\pi}{4}+\frac{X}{4}-\frac{3X^2}{8}-\frac{55X^3}{96}+o\left(X^3\right),$$

and we change X by $\frac{1}{x}$, we obtain

$$\arctan\left(\sqrt{\frac{x+1}{x+2}}\right)=\frac{\pi}{4}-\frac{1}{4x}+\frac{3}{8x^2}-\frac{55}{96x^3}+o\left(\frac{1}{x^3}\right).$$

b) Replaces x bay $\frac{1}{X}$, we obtain $f\left(X\right)=\frac{1}{X^2}\cdot\frac{1-2X^3}{1+X}$. Since the division according to the increasing powers of X, we have

$$\frac{1-2X^3}{1+X}=1-X+X^2-3X^3+3X^4-3X^5+o\left(X^5\right),$$

then

$$f\left(x\right)=x^2\left(1-\frac{1}{x}+\frac{1}{x^2}-\frac{3}{x^3}+\frac{3}{x^4}-\frac{3}{x^5}+o\left(\frac{1}{x^5}\right)\right),$$

therefore

$$f\left(x\right)=x^2-x+1-\frac{3}{x}+\frac{3}{x^2}-\frac{3}{x^3}+o\left(\frac{1}{x^3}\right).$$

8.7 Applications of limited expansion

8.7.1 Calcul of certain limits

One important application of limited expansion (expansion function) is to find the certain limits of a function. The limited expansion leads to a method for give an equivalence form for a given function. This equivalence form leads to evaluating limits of indeterminate forms. This often finds real world applications in problems such as the following.

Example 39. 1) Calculate

$$\lim_{x\to 0}\frac{\cosh\left(x\right)-\frac{12+5x^2}{12-5x^2}}{\sin^2\left(x\right)}.$$

Near 0, we obtained the indeterminate form $\left(\frac{0}{0}\right)$. We will breaks (lift) this form, we can rewrite $x\mapsto\cosh\left(x\right)$, $x\mapsto\sin^2\left(x\right)$ in equivalence forms as

$$\cosh\left(x\right)=1+\frac{x^2}{2}+o\left(x^2\right),$$

and

$$\sin^2(x) = x^2 + o\left(x^2\right).$$

By making the division according to nthe increasing power, we obtain

$$\frac{12+5x^2}{12-5x^2} = 1 + \frac{5}{6}x^2 + o\left(x^2\right).$$

So,

$$\cosh(x) - \frac{12+5x^2}{12-5x^2} = -\frac{x^2}{3} + o\left(x^2\right).$$

Near 0, the quotient rewrite as

$$\frac{\cosh(x) - \frac{12+5x^2}{12-5x^2}}{\sin^2(x)} = \frac{-\frac{x^2}{3} + o\left(x^2\right)}{x^2 + o\left(x^2\right)} = -\frac{1}{3} + o(1).$$

Consequently,

$$\lim_{x\to 0} \frac{ch(x) - \frac{12+5x^2}{12-5x^2}}{\sin^2(x)} = -\frac{1}{3}.$$

8.7.2 Geometric applications

Proposition 28. *Let f is a function defined nearv 0. Suppose that f has a limited expansion near 0 of the form*

$$f(x) = a_0 + a_1 x + a_p x^p + o\left(x^p\right),\ p \geq 2 \text{ and } a_p \neq 0.$$

Then, the graph of f admit at the point $x = 0$, an tangent line to the curve at this point whose equation is

$$y = a_0 + a_1 x.$$

further, if $a_p > 0$, there is a sub-neighbohood 0 in which the curve is above the tangent when p is even. On the other hand, if p is odd, we have a part of the curve above and the other below the tangent

Proposition 29. *Let f an function defined near $+\infty$. Suppose that $\frac{f(x)}{x}$ admits a limited expansion near $+\infty$ of the form*

$$\frac{f(x)}{x} = A + \frac{B}{x} + \frac{C}{x^p} + o\left(\frac{1}{x^p}\right),\ p \geq 2 \text{ and } C \neq 0.$$

Then, the curve of f admits near $+\infty$, an oblique asymptotic of equation

$$y = Ax + B.$$

In addition, the sign of C determine the position of the curve of f with respect to the asymptote. Indeed, if $C > 0$ (respectively $C < 0$) near $+\infty$the curve of f is above (respectively below) oh the asymptote.

Exercise 43. Let f the function defined by $f(x) = x^{\frac{1}{x}}$ and (C_f) it's curve.

1) Calculate the limited expansion of f to the order 2 near 1.

2) Deduce the equation of the tangent of (C_f) at point a of abscissa 1 and the relative position of the curve with respect to the tangent near a.

Solution: 1) By making the change of variable $t = x - 1$ tends to 0 when x tends to 1, we obtain

$$f(t) = (1+t)^{\frac{1}{1+t}} = e^{\frac{\ln(1+t)}{1+t}}.$$

We calculate the limited expansion of the function $t \longmapsto e^{\frac{\ln(1+t)}{1+t}}$ near 0 to the order 2, we obtain

$$f(t) = 1 + t - t^2 + o\left(t^2\right).$$

So, the limited expansion of the function f near 1 to the order 2, is

$$f(x) = 1 + (x-1) - (x-1)^2 + o\left((x-1)^2\right).$$

2) Note that the limited expansion of the function f near 1 to the order 1, allows to obtain the equation of the tangent to (C_f) at point a of abscissa 1. Indeed,

$$f(x) = x + o((x-1)).$$

Thus, the equation of the tangent to (C_f) at point a of abscissa 1is

$$y = x.$$

The sign of $f(x) - y$ will determine the relative position of curve (C_f) with respect to the tengent near 1, we have

$$f(x) - y = -(x-1)^2 + o\left((x-1)^2\right) < 0,$$

which implies that the curve (C_f) below the tangent at point a of abscissa 1.

Exercise 44. Let f the function defined by $f(x) = \frac{x+2}{x}\sqrt{x^2-1}$ and (C_f) it's curve.
1) Calculate the limited expansion of f to the order 2 near $-\infty$.
2) Deduce the asymptote of f when x tends to $-\infty$.
3)) Specify the position of the curve (C_f) with respect to the asymptote near $-\infty$.

Solution: 1)By making the change of variable $x = \frac{1}{X}$ we come back to the limited expansion of $f\left(\frac{1}{X}\right)$ to the 2 near 0^-. We have

$$f(x) = f\left(\frac{1}{X}\right) = (1+2X)\frac{\sqrt{1-X^2}}{|X|},$$

and since $X < 0$, then

$$f\left(\frac{1}{X}\right) = \frac{-1-2X}{X}\sqrt{1-X^2}.$$

By taking the limited expansion of the function $X \longmapsto \sqrt{1-X^2}$ to the order 2 near 0, we obtain

$$\begin{aligned} f\left(\frac{1}{X}\right) &= \frac{-1-2X}{X}\left(1 - \frac{1}{2}X^2 + o\left(X^2\right)\right) \\ &= -2 - \frac{1}{X} - \frac{1}{2}X + o(X). \end{aligned}$$

So, the limite expansion of the function f to the order 2 near $-\infty$, we have

$$= -2 - x - \frac{1}{2x} + o\left(\frac{1}{x}\right).$$

2) We deduce that near $-\infty$, the curve (C_f) admits as asymptote of equation

$$y = -2 - x.$$

3) The relative position of the (C_f) with respect to the asymptote near $-\infty$. We study the sign of

$$f(x) - y = f(x) + 2 + 2x = \frac{1}{2x} o\left(\frac{1}{x}\right) < 0,$$

when $x \to -\infty$ which is that of $\frac{1}{2x}$, then the curve below of the asymptote.

Chapter 9

Definite and indefinite Integrals (Calcul of primitives (antiderivatives)

9.1 Indefinite and definite integrals

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f, how can we find a function with derivative f? If we can find a function F derivative f we call F an antiderivative of f.

Definition 41. If f be a function defined and continuous on $I \subset \mathbb{R}$ and if there is a function F defirentiable on I such that $F'(x) = f(x), \ \forall x \in I$, we say that F is an antiderivative (primitive) of f on I (F is a said to be an antiderivative (primitive) of f on I).

Proposition 30. *If F and G are two primitives of f on I, then $F-G$ is a constant function.*

Example 40. 1) La fonction $F(x) = x^2 + c$, où $c \in \mathbb{R}$ est une primitive de $f(x) = 2x$ sur $\mathbb{R}$.

2) The function $F(x) = \ln(x) + c$, where $c \in \mathbb{R}$(where c is an arbitrary constant.) is a primitive of $f(x) = \frac{1}{x}$ on $]0, +\infty[$.

we say that $F(x) + c$ is the most general antiderivative of f and write

$$\int f(x)\,dx = F(x) + c, \ c \in \mathbb{R},$$

where the symbol $\int$ is called an integral sign, and $\int f(x)\,dx$ is called the indefinite integral of f.

The next two results are versions of the most elementary form of the fundamental theorem of calculus :

Definition 42. Let f is a continuous function on open interval I and $a \in I$. The function F defined on I by

$$F(x) = \int_a^x f(t)\,dt,$$

is a antiderivative (primitive) of f on I. Where the right-hand side $\int_a^x f(t)\,dt$, represents the definite integral of the function f from a to x.

Theorem 25. *If f be a continuous function on $[a,b]$ and if F is an antiderivative (primitive) of f on $[a,b]$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

wich denoted by $[F(x)]_a^b = F(b) - F(a)$.

Proposition 31. *If f be a bounded integrably function bornée intégrable on $[a,b]$, then the function F defined by*

$$F(x) = \int_a^x f(t)\,dt,$$

is continuous on $[a,b]$.

9.2 Fundamental integrals (Integrals for refference - Calculating of primitives)

There are sophisticated methods developed to find antiderivatives of various kinds of functions. We will study some of these later.

We shall now give a list of formula, which may be regarded as fundamental, and to which all integrals must ultimately be reduced. We shall then consider in this chapter such examples as are integrable by these formula, either directly, or after some simple transformation.

We give for reference a list of some of the integrals the preceding chapters : Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions.

Function f	Primitive F
$a,\ (a \in \mathbb{R})$	$ax + c,\ (c \in \mathbb{R})$
$x^n, \quad (n \in \mathbb{R} - \{-1\})$	$\frac{x^{n+1}}{n+1} + c, \quad (c \in \mathbb{R})$
$\frac{1}{x}, \quad x > 0$	$\ln(x) + c, \quad (c \in \mathbb{R})$
$\cos(x)$	$\sin(x) + c, \quad (c \in \mathbb{R})$
$\sin(x)$	$-\cos(x) + c, \quad (c \in \mathbb{R})$
$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$	$\tan(x) + c, \quad (c \in \mathbb{R})$
e^x	$e^x + c, \quad (c \in \mathbb{R})$
$\cosh(x)$	$\sinh(x) + c, \quad (c \in \mathbb{R})$
$\sinh(x)$	$\cosh(x) + c, \quad (c \in \mathbb{R})$
$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$	$\tanh(x) + c, \quad (c \in \mathbb{R})$
$\frac{1}{\sqrt{1-x^2}}, \quad \|x\| < 1$	$\arcsin(x) + c = -\arccos(x) + c', \quad (c,\ c' \in \mathbb{R})$
$\frac{1}{\sqrt{1+x^2}}$	$\arg\sinh(x) + c, \quad (c \in \mathbb{R})$
$\frac{1}{\sqrt{x^2-1}}, \quad x > 1$	$\arg\cosh(x) + c, \quad (c \in \mathbb{R})$
$\frac{1}{1+x^2}$	$\arctan(x) + c, \quad (c \in \mathbb{R})$
$\frac{1}{1-x^2}$	$\frac{1}{2}\ln\left(\left\|\frac{1+x}{1-x}\right\|\right) + c, \quad (c \in \mathbb{R})$

Function f	Primitive F
$\frac{1}{\sqrt{x^2-1}}, \quad x > 1$	$\arg\cosh(x) + c, \quad (c \in \mathbb{R})$
$\frac{1}{\sqrt{x^2-1}}, \quad x < -1$	$-\arg\cosh(-x) + c, \quad (c \in \mathbb{R})$
$\frac{1}{1-x^2}, \quad \|x\| < 1$	$\arg\tanh(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + c, \quad (c \in \mathbb{R})$
$\frac{1}{1-x^2}, \quad \|x\| > 1$	$\arg\coth(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) + c, \quad (c \in \mathbb{R})$

9.3 Ruls of calculating

We know that the derivative of the sum or difference of two functions is the sum or difference of their derivatives. This suggests that the antiderivative of the sum or difference of two functions is the sum or difference of their antiderivatives:

9.3.1 Linearity

1) $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$,
2) $\int \lambda f(x)\,dx = \lambda \int f(x)\,dx$, where $\lambda \in \mathbb{R}$.

9.3.2 Substitution ruls (change of variables)

Let I and J two intervals of $\mathbb{R}$, $a,\ b \in I$, $u : I \to J$ be a function of class C^1 and $f : J \to \mathbb{R}$ be a continuous function. Then

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(t)\,dt.$$

Example 41. 1) We will calculate

$$\int \frac{\ln(x)}{x} dx.$$

We make the subtitution $u(x) = \ln(x)$. We have $u'(x) = \frac{1}{x}$, we obtain

$$\int \frac{\ln(x)}{x} dx = \int u(x)\, du(x) = \frac{u^2(x)}{2} = \frac{1}{2}\ln^2(x) + c,\ c \in \mathbb{R}.$$

2) We will calculate

$$\int 2x\sqrt{x^2-1}dx.$$

We make the subtitution $u(x) = x^2 - 1$. On a $u'(x) = 2x$, on obtient

$$\begin{aligned}\int 2x\sqrt{x^2-1}dx = \int u'(x)\sqrt{u(x)}dx = \int \sqrt{u(x)}u'(x)\,dx \\ = \int \sqrt{u(x)}du(x) \\ = \frac{2}{3}(u(x))^{\frac{3}{2}} + c = \frac{2}{3}\sqrt{(x^2-1)^3} + c,\ c \in \mathbb{R}.\end{aligned}$$

3) We will calculate

$$\int \sin(x)\cos(x)\,dx.$$

We make the subtitution $u(x) = \sin(x)$. On a $u'(x) = \cos(x)$, on obtient

$$\int \frac{\cos(x)}{\sin(x)}dx = \int \frac{u'(x)}{u(x)}dx = \ln|u(x)| + c = \ln|\sin(x)| + c, \quad c \in \mathbb{R}.$$

4) We will calculate

$$\int \frac{\cos(x)}{\sin(x)}dx.$$

We make the subtitution $u(x) = \sin(x)$. On a $u'(x) = \cos(x)$, on obtient

$$\int \sin(x)\cos(x)\,dx = \int u(x)\,u'(x)\,dx = \frac{1}{2}u^2(x) + c = \frac{1}{2}\sin^2(x) + c, \quad c \in \mathbb{R}.$$

5) We will calculate

$$\int \frac{1}{a+x^2}dx,\ a \in \mathbb{R}.$$

We have (The a's separate out)

$$\int \frac{1}{a+x^2}dx = \int \frac{1}{a\left(1+\frac{x^2}{a}\right)}dx = \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{\sqrt{a}}\right)^2}dx.$$

We make the subtitution $u(x) = \frac{x}{\sqrt{a}}$. We have $u'(x) = \frac{1}{\sqrt{a}}$, we obtain (The a's separate out)

$$\int \frac{1}{a+x^2}dx = \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{\sqrt{a}}\right)^2}$$

$$= \frac{\sqrt{a}}{a}\int \frac{\frac{1}{\sqrt{a}}}{1+\left(\frac{x}{\sqrt{a}}\right)^2}dx$$

$$= \frac{\sqrt{a}}{a}\int \frac{du}{1+(u)^2} = \frac{\sqrt{a}}{a}\arctan(u) + c,$$

$$= \frac{\sqrt{a}}{a}\arctan\left(\frac{x}{\sqrt{a}}\right) + c, \quad c \in \mathbb{R}.$$

6) We will calculate

$$\int \frac{x^3}{x^8-2}dx.$$

We make the subtitution $u(x) = x^4$. We have $u'(x) = 4x$, we obtain

$$\int \frac{x^3}{(x^4)^2-2}dx = \frac{1}{4}\int \frac{u'(x)}{u^2(x)-2}dx$$

$$= \frac{1}{8}\int \frac{du}{\left(\frac{u}{\sqrt{2}}\right)^2-1}$$

$$= \frac{\sqrt{2}}{8}\int \frac{1}{\left(\frac{u}{\sqrt{2}}\right)^2-1}\frac{du}{\sqrt{2}}$$

$$= -\frac{\sqrt{2}}{8}\int \frac{1}{1-\left(\frac{u}{\sqrt{2}}\right)^2}\frac{du}{\sqrt{2}}$$

$$= \begin{cases} -\frac{\sqrt{2}}{8}\operatorname{arg\,tanh}\left(\frac{u}{\sqrt{2}}\right) + c, \ if \ \left|\frac{u}{\sqrt{2}}\right| < 1, \\ -\frac{\sqrt{2}}{2}\operatorname{arg\,coth}\left(\frac{u}{\sqrt{2}}\right) + c, \ if \ \left|\frac{u}{\sqrt{2}}\right| > 1. \end{cases}$$

$$= \begin{cases} -\frac{\sqrt{2}}{8}\operatorname{arg\,tanh}\left(\frac{x^4}{\sqrt{2}}\right) + c, \ if \ \left|\frac{x^4}{\sqrt{2}}\right| < 1, \\ -\frac{\sqrt{2}}{8}\operatorname{arg\,coth}\left(\frac{x^4}{\sqrt{2}}\right) + c, \ if \ \left|\frac{x^4}{\sqrt{2}}\right| > 1. \end{cases}$$

6) We will calculate

$$\int \sqrt{1-x^2}dx.$$

We make the subtitution $x = \sin(u)$. We have $dx = \cos(u)\,du$, we obtain, for $u(x) > 0$,

$$\int \sqrt{1-x^2}dx = \int \sqrt{1-\sin^2(u)}\cos(u)\,du$$

$$= \int \cos^2(u)\,du$$
$$= \int \frac{1+\cos(2u)}{2}du$$
$$= \frac{1}{2}\left(u + \frac{1}{2}\sin(2u)\right) + c.$$
$$= \frac{1}{2}\left(u + \cos(u)\sin(u)\right) + c$$
$$= \frac{1}{2}\left(\arcsin(x) + x\sqrt{1-x^2}\right) + c,\ c \in \mathbb{R}.$$

Exercise 45. Calculate (Compute) the following les integrals :
1) $\int \frac{x}{\sqrt{1-x^2}}dx$
2) $\int \frac{e^x}{2+e^{2x}}dx$
3) $\int \frac{e^x}{\left(\frac{1}{2}+e^x\right)^2}dx$

Solution: 1) We make the subtitution $u = x^2$. We have $u' = 2x$, we obtain

$$\int \frac{x}{\sqrt{1-x^2}}dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u}} = -\sqrt{1-u} + c = -\sqrt{1-x^2} + c,\ c \in \mathbb{R}.$$

2) We make the subtitution $u = e^x$. We have $u' = e^x$, we obtain

$$\int \frac{e^x}{2+e^{2x}}dx = \int \frac{du}{2+u^2} = \frac{1}{2}\int \frac{du}{1+\left(\frac{u}{\sqrt{2}}\right)^2} = \frac{\sqrt{2}}{2}arctan\left(\frac{u}{\sqrt{2}}\right).$$

3) We make the subtitution $u = \frac{1}{2} + e^x$. We have $u' = e^x$, we obtain

$$\int \frac{e^x}{\left(\frac{1}{2}+e^x\right)^2}dx = \int \frac{du}{u^2} = -\frac{1}{u} + c = -\frac{1}{\frac{1}{2}+e^x} + c, \quad c \in \mathbb{R}.$$

9.3.3 Integration by part

Let u and v two functions of a class $C^1[a,b]$. Then

$$\int_a^b v(x)\,u'(x)\,dx = [u(x)\,v(x)]_a^b - \int_a^b u(x)\,v'(x)\,dx.$$

The key is in choosing u and v. The goal of that choice is to make $\int v(x)\,dx$ easier than $\int u(x)\,dx$. This is best seen by examples.

Example 42. 1) We will calculate

$$\int \ln(x)\,dx.$$

Put

$$u(x) = \ln(x), \quad v'(x) = 1,$$

then

$$u'(x) = \frac{1}{x}, \quad v(x) = x.$$

By using the integration by part, we have

$$\int \ln(x)\,dx = x\ln(x) - \int \frac{1}{x}dx = x\ln(x) - \ln(x) + c,\ c \in \mathbb{R}.$$

2) We will calculate

$$\int \arctan(x)\,dx.$$

Put

$$u(x) = \arctan(x), \quad v'(x) = 1,$$

then

$$u'(x) = \frac{1}{x^2+1}, \quad v(x) = x.$$

By using the integration by part, we have

$$\int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{x^2+1}dx = x\arctan(x) - \frac{1}{2}ln\left(x^2+1\right) + c,\ c \in \mathbb{R}.$$

3) We will calculate

$$\int x^2\cos(2x)\,dx.$$

Put

$$u(x) = x^2, \quad v'(x) = \cos(2x),$$

then

$$u'(x) = 2x, \quad v(x) = \frac{1}{2}\sin(2x).$$

By using the integration by part, we have

$$\int x^2\cos(2x)\,dx = \frac{1}{2}x^2\sin(2x) - \int x\sin(2x)\,dx.$$

By using the integration by part for the second time, we put

$$u(x) = x, \quad v'(x) = \sin(2x),$$

then

$$u'(x) = 1, \quad v(x) = -\frac{1}{2}\cos(2x),$$

hence

$$\int x\cos(2x)\,dx = -\frac{1}{2}x\cos(2x)+\frac{1}{2}\int \cos(2x)\,dx = -\frac{1}{2}x\cos(2x)+\frac{1}{4}\sin(2x)+c,\ c\in\mathbb{R}.$$

Therefore

$$\begin{aligned}\int x^2\cos(2x)\,dx &= \frac{1}{2}x^2\sin(2x)-\int x\sin(2x)\,dx\\ &= \frac{1}{2}x^2\sin(2x)+\frac{1}{2}x\cos(2x)-\frac{1}{4}\sin(2x)+c,\ c\in\mathbb{R}.\end{aligned}$$

3) We will calculate

$$\int x^2e^x dx.$$

Put

$$u(x)=x^2,\quad v'(x)=e^x,$$

then

$$u'(x)=2x,\quad v(x)=e^x.$$

By using the integration by part, we have

$$\int x^2e^x dx = x^2e^x-\frac{1}{2}\int xe^x dx.$$

By using the integration by part for the second time, we put

$$u(x)=x,\quad v'(x)=e^x,$$

then

$$u'(x)=1,\quad v(x)=e^x,$$

hence

$$\int xe^x dx = xe^x-\int e^x dx = xe^x-e^x+c,\ c\in\mathbb{R}.$$

Therefore

$$\begin{aligned}\int x^2e^x dx &= x^2e^x-\int xe^x dx\\ &= \left(x^2-x+1\right)e^x+c,\ c\in\mathbb{R}.\end{aligned}$$

9.3.4 Techniques of integration

9.3.4.1 Integration of fractional fractions :

To find the primitive of f where $f(x)=\frac{p(x)}{q(x)}$, that is a rational fraction is integrated by decomposing it into partial fractions, whose denominators are the factors of the original denominator. The complete discussion of partial fractions belongs to Algebra. We shall only consider here the form of these partial fractions and the processes of determining them. Or we make the numerator $p(x)$ the differential of the denominator $q(x)$ (i.e., we will appear the differential of the denomerator in numurator). Two types of partial fractions will apear:

1) $\frac{1}{(x-a)^n}$, $n\in\mathbb{N}$, $a\in\mathbb{R}$. For $n=1$(see, subSection 9.3.2. Example 33. 5.)

2) $\frac{ax+b}{(x^2+cx+d)^n}$, $n\in\mathbb{N}$, where the polynomial x^2+cx+d hasn't real roots ($\Delta=c^2-4d<0$) or has two real roots ($\Delta=c^2-4d>0$).

Preliminary operation : 1) If the degree of the numerator is equal to, or greater than , that o f the denominator, the fraction should be reduced to a mixed quantity, by dividing the numerator by the denominator. The degree o f the numerator o f this new fraction will be less than that o f the denominator. Such fractions only will be considered in the following articles :
2) Factors of the denominator : A rational fraction is integrated by decomposing it into partial fractions,whose denominators are the factors o f the original denominator. Now it is shown by the theory of equations, that a polynomial o f the n^{th} degree with respect to as, may be resolved into n factors o f the first degree or factors of the denominator all of the first degree, and some repeated (common denominator). **These factors are real or imaginary, but the imaginary factors will occur in pairs, of the form**

$$x - a + b\sqrt{-1},\ or\ x - a - b\sqrt{-1},\ where\ i^2 = -1,$$

whose product is $(x-a)^2 + b^2$ **a real factor of the second degree.**

Remark 29. It follows that any polynomial may be resolved into real factors of the first or second degree, and only such factors will be considered in the denominators of fractions. There are four cases to be considered.
First. Where the denominator contains factors of the first degree only, each of which occurs but once.
Second. Where the denominator contains factors of the first degree only, some of which are repeated.
Third. Where the denominator contains factors of the second degree, each of which occurs but once.
Fourth. Where the denominator contains factors of the second degree, some of which are repeated.

Example 43. 1) We will calculate

$$\int \frac{x^3 - 2x^2}{x^3+1} dx = \int \frac{x^3 + 1 - 1 - 2x^2}{x^3+1} dx$$
$$= \int dx - \int \frac{2x^2+1}{x^3+1} dx$$

2) By the division

$$\int \frac{2x^5 - 3x^4 + 1}{x^4 + x^3} dx = \int \frac{x^3 + 1 - 1 - 2x^2}{x^3+1} dx$$
$$= \int (2x-3)\, dx - \int \frac{2x^3 - 3x^2 + 1}{x^4+x^3} dx.$$

3) We will calculate

$$\int \frac{x^3+1}{x(x-1)^3} dx = -\int \frac{1}{x} dx + \int \frac{2}{x-1} dx + \int \frac{1}{(x-1)^2} dx + \int \frac{2}{(x-1)^3} dx$$
$$= -\ln(x) + 2\ln|x-1| - \frac{1}{(x-1)^2} - \frac{1}{x-1} + c.$$

4) We will calculate

$$\int \frac{x^3+6x+8}{x^3-4x}dx.$$

The given fraction may be decomposed into partial fractions (break the fraction into three terms), we have

$$\int \frac{x^3+6x+8}{x^3-4x}dx = \int \left(\frac{1}{x-2} - \frac{2}{x+2} + \frac{1}{x}\right) dx$$
$$= \ln|x-2| - 2\ln|x+2| + \ln|x| + c.$$

5) We will calculate

$$\int \frac{1}{(x^2+a^2)^2}dx.$$

To integrate the last fraction , we use the following formula of reduction

$$\int \frac{1}{(x^2+a^2)^n}dx = \frac{1}{2(n-1)a}\left[\frac{x}{(x^2+a^2)^{n-1}} + (2n-3)\int \frac{1}{(x^2+a^2)^{n-1}}dx\right].$$

To apply this formula to

$$\int \frac{1}{(x^2+1)^2}dx,$$

we make $a=1$ and $n=2$, we have

$$\int \frac{1}{(x^2+1)^2}dx = \frac{1}{2}\left[\frac{x}{x^2+1} + \int \frac{1}{x^2+1}dx\right]$$

$$\int \frac{1}{(x^2+1)^2}dx = \frac{1}{2}\frac{x}{x^2+1} + \arctan(x) + c, \quad c \in \mathbb{R}.$$

Second method :

$$\int \frac{1}{(x^2+a^2)^2}dx = \frac{1}{a^2}\int \frac{a^2}{(x^2+a^2)^2}dx$$
$$= \frac{1}{a^2}\int \frac{a^2+x^2-x^2}{(x^2+a^2)^2}dx$$
$$= \frac{1}{a^2}\int \frac{a^2+x^2}{(x^2+a^2)^2}dx - \frac{1}{a^2}\int \frac{x^2}{(x^2+a^2)^2}dx$$
$$= \frac{1}{a^2}\int \frac{1}{x^2+a^2}dx - \frac{1}{a^2}\int \frac{x^2}{(x^2+a^2)^2}dx$$
$$= \frac{1}{a^3}\arctan\left(\frac{x}{a}\right) - \frac{1}{a^2}\int \frac{x^2}{(x^2+a^2)^2}dx.$$

To calculate $\int \frac{x^2}{(x^2+a^2)^2}dx = \int \frac{2x}{(x^2+a^2)^2}\frac{1}{2}xdx$, we use the integration by part, we set

$$u(x) = \frac{1}{2}x, \quad v'(x) = \frac{2x}{(x^2+a^2)^2},$$

then

$$u'(x) = \frac{1}{2}, \quad v(x) = \frac{-1}{x^2+a^2},$$

Therefore

$$\begin{aligned}\int \frac{x^2}{(x^2+a^2)^2}dx &= \int \frac{2x}{(x^2+a^2)^2}\frac{1}{2}xdx \\ &= \frac{-x}{2(x^2+a^2)} + \frac{1}{2}\int \frac{1}{x^2+a^2}dx \\ &= \frac{-x}{2(x^2+a^2)} + \frac{1}{2a}\arctan\left(\frac{x}{a}\right) + c.\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{1}{(x^2+a^2)^2}dx &= \frac{1}{a}\arctan\left(\frac{x}{a}\right) - \frac{1}{a^2}\int \frac{x^2}{(x^2+a^2)^2}dx \\ &= \frac{1}{a^3}\arctan\left(\frac{x}{a}\right) + \frac{x}{2(x^2+a^2)} - \frac{1}{2a^3}\arctan\left(\frac{x}{a}\right) + c \\ &= \frac{1}{2a^3}\arctan\left(\frac{x}{a}\right) + \frac{x}{2(x^2+a^2)} + c, \quad c \in \mathbb{R}\end{aligned}$$

2) We will calculate

$$\int \frac{x+1}{x^2-2x+5}dx.$$

We make to appear the derivative $2x-2$ of the polynomial x^2-2x+5 in numerator. We have

$$\begin{aligned}\int \frac{x+1}{x^2-2x+5}dx &= \frac{1}{2}\int \frac{2x+2}{x^2-2x+5}dx \\ &= \frac{1}{2}\int \frac{2x+2}{x^2-2x+5}dx \\ &= \frac{1}{2}\int \frac{2x+2-2+2}{x^2-2x+5}dx \\ &= \frac{1}{2}\int \frac{2x-2}{x^2-2x+5}dx + \frac{1}{2}\int \frac{4}{x^2-2x+5}dx \\ &= \frac{1}{2}\int \frac{2x-2}{x^2-2x+5}dx + \frac{1}{2}\int \frac{4}{(x-1)^2+4}dx \\ &= \frac{1}{2}\int \frac{2x-2}{x^2-2x+5}dx + \frac{1}{2}\int \frac{1}{\left(\frac{x-1}{2}\right)^2+1}dx \\ &= \frac{1}{2}ln\left|x^2-2x+5\right| + \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + c, \quad c \in \mathbb{R}.\end{aligned}$$

2) We will calculate

$$\int \frac{x^2+2}{x^3+1}dx.$$

Since the degree of the numerator is less than , that o f the denominator. Then, the rational fraction is integrated by decomposing it into partial fractions. We have

$$x^3 - 1 = (x+1)\left(x^2 - x + 1\right),$$

where $(x^2 - x + 1)$ **is a real factor of the second degree**.
The decompositions into partial fraction of $\frac{x^2+2}{x^3+1}$ is

$$\begin{aligned}\frac{x^2+2}{x^3+1} &= \frac{x^2+2}{(x+1)(x^2-x+1)} \\ &= \frac{a}{x+1} + \frac{bx+c}{x^2-x+1} \qquad (9.1)\end{aligned}$$

Multiplying (9.1) by $x+1$ and take x tends -1, the left-hand side tends to 1 and the second tends to a. Then $a = 1$.
We take (by substuting) $x = 0$ in (9.1), we obtain $a + c = 2$. Then $c = 1$.
Multiplying (9.1) by x and take x tends $+\infty$, the left-hand side tends to 1 and the second tends to $a + b$. Then $b = 0$.
Therefore

$$\frac{x^2+2}{x^3+1} = \frac{1}{x+1} + \frac{1}{x^2-x+1},$$

thus, we integrating

$$\int \frac{x^2+2}{x^3+1}dx = \int \frac{1}{x+1}dx + \int \frac{1}{x^2-x+1}dx.$$

We write the denomunator $x^2 - x + 1$ in the forme (Having written the denominator in the form) $(x-a)^2 + b^2$(called completing the square), we will begin to explain it.
To match the quadratic with the square, we fix up the constant: We have $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2 + 1\right]$, and make the subtutition $u = \frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$, we obtain

$$\begin{aligned}\int \frac{1}{x^2-x+1}dx &= \frac{2}{\sqrt{3}} \int \frac{du}{1+u^2} \\ &= \frac{2}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + c.\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{x^2+2}{x^3+1}dx &= \int \frac{1}{x+1}dx + \int \frac{1}{x^2-x+1}dx. \\ &= \ln|x+1| + \frac{2}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + c.\end{aligned}$$

Exercise 46. Calculate the following integrals
1) -a) $I = \int \frac{x^4}{x^2+4}dx$, -b) $J = \int \frac{1}{(x+2)(x^2-4)}dx$.
2)-a) $I = \int \frac{1}{x^2(x^2+2)^2}dx$, - b) $J = \int \frac{1}{x^2(x^2+2x+1)}dx$.

Solution: The rational fraction is integrated by decomposing it into partial fractions

1)-a) Since the degree of the numerator is equal to, or greater than , that o f the denominator, the fraction should be reduced to a mixed quantity, by dividing the numerator by the denominator. The degree of the numerator of this new fraction will be less than that o f the denominator. Dividing x^4 by x^2-4, we have

$$\begin{aligned} I &= \int \frac{x^4}{x^2+4}dx \\ &= \int \left(x^2+4\right)dx + \int \frac{16}{x^2+4}dx. \end{aligned} \tag{9.2}$$

The form of decomposition of $\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)}$ is

$$\frac{1}{x^2-4} = \frac{a}{x-2} + \frac{b}{x+2},$$

multiplying by $x-2$ and substutite $x=2$, we obtain $a=\frac{1}{4}$,
multiplying by $x+2$ and substutite $x=-2$, we obtain $b=-\frac{1}{4}$,
then

$$\frac{1}{x^2-4} = \frac{1}{4(x-2)} - \frac{1}{4(x+2)}. \tag{9.3}$$

From (9.2) and (9.3), we deducde

$$\begin{aligned} I &= \int \left(x^2+4\right)dx + 4\int \frac{1}{x-2}dx - 4\int \frac{1}{x+2}dx \\ &= \frac{1}{3}x^3 + 4x + 4ln\left|x-2\right| - 4ln\left|x-2\right| + c, \quad c\in\mathbb{R}. \end{aligned}$$

b) The form of decomposition of $\frac{1}{(x+2)(x^2-4)} = \frac{1}{(x-2)(x+2)^2}$ is

$$\frac{1}{(x-2)(x+2)^2} = \frac{a}{x-2} + \frac{b}{x+2} + \frac{c}{(x+2)^2},$$

multiplying by $x-2$ and substutite $x=2$, we obtain $a=\frac{1}{16}$,
multiplying by $(x+2)^2$ and substutite $x=-2$, we obtain $c=-\frac{1}{4}$,
multiplying by x and take x tends $+\infty$, we obtain $a+b+c=0$, then $b=-a=-\frac{1}{16}$.
Thus

$$\frac{1}{(x+2)(x^2-4)} = \frac{1}{16(x-2)} - \frac{1}{4(x+2)} - \frac{1}{4(x+2)^2}.$$

Therefore

$$\begin{aligned} J &= \int \frac{1}{(x+2)(x^2-4)}. \\ &= \frac{1}{16}\int \frac{1}{x-2}dx - \frac{1}{16}\int \frac{1}{x+2}dx - \frac{1}{4}\int \frac{1}{(x+2)^2}dx \\ &= \frac{1}{16}\ln\left|x-2\right| - \frac{1}{16}\ln\left|x+2\right| + \frac{1}{4(x+2)} + c, \quad c\in\mathbb{R}. \end{aligned}$$

2)-a) Similarly, we obtain

$$\frac{1}{x^2\left(x^2+2\right)^2}=\frac{1}{4x^2}-\frac{1}{4\left(x^2+2\right)}-\frac{1}{2\left(x^2+2\right)^2}.$$

Integrating , we obtain

$$\begin{aligned} I=\int\frac{1}{x^2\left(x^2+2\right)^2}dx &= \frac{1}{4}\int\frac{1}{x^2}dx-\frac{1}{4}\int\frac{1}{x^2+2}dx-\frac{1}{2}\int\frac{1}{\left(x^2+2\right)^2}dx \\ &= -\frac{1}{4x}-\frac{x}{8\left(x^2+2\right)}dx-\frac{3}{8\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)+c, \quad c\in\mathbb{R}. \end{aligned}$$

Remark 30. To calculate $\int\frac{1}{(x^2+2)}dx$ see subSection 9.3.4.1, example 15. 1.

b) The decomposition form of $\frac{1}{x^2(2x^2+2x+1)}$ is

$$\frac{1}{x^2\left(2x^2+2x+1\right)^2}=\frac{a}{x}+\frac{b}{x^2}+\frac{cx+d}{2x^2+2x+1},$$

multiplying by x^2 and substutite $x=0$, we obtain $b=1$,
the root of $2x^2+2x+1$ is $\alpha=-\frac{1}{2}+\frac{1}{2}i$, multiplying by $2x^2+2x+1$ and substutite $x=\alpha$, we obtain

$$\begin{aligned} c\alpha+d=\frac{1}{\alpha^2}=2i &\Rightarrow -\frac{c}{2}+d+\frac{c}{2}i=2i \\ &\Rightarrow -\frac{c}{2}+d+\frac{c}{2}i=2i \\ &\Rightarrow \begin{cases} c=4, \\ d=2. \end{cases} \end{aligned}$$

Multiplying by x and take x tends $+\infty$, we obtain $a+\frac{c}{2}=0$, then $a=-2$. Thus

$$\frac{1}{x^2\left(2x^2+2x+1\right)^2}=-\frac{2}{x}+\frac{1}{x^2}+\frac{4x+2}{2x^2+2x+1}.$$

Integrating , we obtain

$$\begin{aligned} \int\frac{1}{x^2\left(2x^2+2x+1\right)} &= -2\int\frac{1}{x}dx+\int\frac{1}{x^2}dx+\int\frac{4x+2}{2x^2+2x+1}dx \\ &= -2\ln|x|-\frac{1}{x}+\ln|2x+2x+1|+c,\ c\in\mathbb{R}. \end{aligned}$$

9.3.5 Integraral of $\frac{ax+b}{\sqrt{x^2+cx+d}}$ and $\sqrt{x^2+cx+d}$

In the first integral, we will make to appear the derivative of x^2+cx+d in numerator. In the second we completing the squar : $x^2+cx+d=\left(x+\frac{c}{2}\right)^2\pm\frac{c^2-4d}{4}$.

Example 44. 1) Find an primitive of $\frac{x}{\sqrt{x^2+2x-3}}$, $\forall x \in]-\infty, -3[\cup]1, +\infty[$.
we will make to appear the derivative $2x+2$ of the polynomial x^2+2x-3. We have

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2+2x-3}}dx &= \frac{1}{2}\int \frac{2x}{\sqrt{x^2+2x-3}}dx \\
&= \frac{1}{2}\int \frac{2x+2-2}{\sqrt{x^2+2x-3}}dx \\
&= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x-3}}dx - \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x-3}}dx \\
&= \int \frac{2x+2}{2\sqrt{x^2+2x-3}}dx + \int \frac{1}{\sqrt{(x+1)^2-4}}dx \\
&= \sqrt{x^2+2x-3} + \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{x-1}{2}\right)^2-1}}dx \\
&= \sqrt{x^2+2x-3} + \ln\left|\left(\frac{x+1}{2}\right) + \sqrt{\left(\frac{x+1}{2}\right)^2-1}\right| + c, \quad c \in \mathbb{R}.
\end{aligned}$$

2) Find an primitive of $\sqrt{x^2+2x-3}$, $\forall x \in]-\infty, -3[\cup]1, +\infty[$.
we completing the squar x^2+2x-3. We have $x^2+2x-3 = (x+1)^2-4$, then

$$\begin{aligned}
\int \sqrt{x^2+2x-3}dx &= \int \sqrt{(x+1)^2-4}dx \\
&= 2\int \sqrt{\left(\frac{x+1}{2}\right)^2-1}dx.
\end{aligned}$$

We make the subtutiton $u = \frac{x+1}{2} \Rightarrow du = \frac{1}{2}dx$. We have

$$\begin{aligned}
\int \sqrt{x^2+2x-3}dx &= 2\int \sqrt{\left(\frac{x+1}{2}\right)^2-1}dx \\
&= \int \sqrt{u^2-1}du.
\end{aligned}$$

Integrating by part, we set

$$f(u) = \sqrt{u^2-1}, \quad g'(u) = 1,$$

then

$$f'(u) = \frac{2u}{2\sqrt{u^2-1}}, \quad g(u) = u,$$

Thus

$$\int \sqrt{u^2-1}du = u\sqrt{u^2-1} - \int \frac{u^2}{\sqrt{u^2-1}}du$$

$$
\begin{aligned}
&= u\sqrt{u^2-1} - \int \frac{u^2-1+1}{\sqrt{u^2-1}}du \\
&= u\sqrt{u^2-1} - \int \frac{u^2-1}{\sqrt{u^2-1}}du - \int \frac{1}{\sqrt{u^2-1}}du \\
&= u\sqrt{u^2-1} - \int \sqrt{u^2-1}du - \int \frac{1}{\sqrt{u^2-1}}du.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\int \sqrt{u^2-1}du &= \frac{1}{2}u\sqrt{u^2-1} - \frac{1}{2}\int \frac{1}{\sqrt{u^2-1}}du \\
&= \frac{1}{2}u\sqrt{u^2-1} - \frac{1}{2}\ln\left|u+\sqrt{u^2-1}\right| + c \\
&= \frac{1}{2}\left(\frac{x+1}{2}\right)\sqrt{\left(\frac{x+1}{2}\right)^2-1} - \frac{1}{2}\ln\left|\frac{x+1}{2}+\sqrt{\left(\frac{x+1}{2}\right)^2-1}\right| + c,\ c\in\mathbb{R}.
\end{aligned}
$$

Exercise 47. Calculate the following integrals
1) $I=\int\sqrt{4x-x^2}dx,\ J=\int\sqrt{x^2+4x+5}dx,$
2) $I=\int\frac{1}{(4x-x^2)^{\frac{3}{2}}}dx,\ J=\int\frac{1}{\sqrt{x^2+4x+5}}dx,$

Solution: We write the quadratic form for $4x-x^2$. we have $4x-x^2=4-(x-2)^2$. Then

$$
\begin{aligned}
I &= \int \sqrt{4x-x^2}dx \\
&= \int \sqrt{4-(x-2)^2}dx \\
&= 2\int \sqrt{1-\left(\frac{x-2}{2}\right)^2}dx.
\end{aligned}
$$

We make the subtitution $x-2=2\cos(t)$, $\quad t\in\,]0,\pi[$, we obtain

$$
\begin{aligned}
I &= 2\int \sqrt{1-\left(\frac{x-2}{2}\right)^2}dx \\
&= 2\int \sqrt{1-\sin^2(t)}\,(-2\sin(t))\,dt \\
&= -4\int \cos(t)\sin(t)\,dt \\
&= -2\int \sin(2t)\,dt \\
&= \cos(2t)+c
\end{aligned}
$$

$$= \cos\left(2\arcsin\left(\frac{x-2}{2}\right)\right) + c,\ c \in \mathbb{R}.$$

2)- a) We have

$$I = \int \frac{1}{(4x - x^2)^{\frac{3}{2}}} dx$$

$$= \int \frac{1}{\left(\sqrt{4x - x^2}\right)^3} dx$$

$$= \int \frac{1}{\left(\sqrt{4 - (x-2)^2}\right)^3} dx$$

$$= \frac{1}{2} \int \frac{1}{\left(\sqrt{1 - \left(\frac{x-2}{2}\right)^2}\right)^3} dx.$$

We make the subtitution $x - 2 = 2\cos(t)$, we obtain

$$I = \frac{1}{8} \int \frac{1}{\left(\sqrt{1 - \cos^2(t)}\right)^3} (-2\sin(t))\, dt$$

$$= \frac{1}{8} \int \frac{1}{\sin^3(t)} (-2\sin(t))\, dt$$

$$= \frac{1}{4} \int -\frac{1}{\sin^2(t)} dt$$

$$= \frac{1}{4} \cot(t) + c$$

$$= \frac{1}{4} \frac{\cos(t)}{\sin(t)} + c$$

$$= \frac{x-2}{4\sqrt{4x - x^2}} + c,\ c \in \mathbb{R}.$$

-b) We write the quadratic form for $x^2 + 4x + 5$. We have $x^2 + 4x + 5 = (x+2)^2 + 1$. Then

$$I = \int \sqrt{x^2 + 4x + 5} dx$$

$$= \int \sqrt{(x+2)^2 + 1} dx$$

We make the subtitution $x + 2 = \sinh(t)$, $t \in \mathbb{R}$, we obtain

$$I = \int \sqrt{x^2 + 4x + 5} dx$$

$$= \int \sinh(t) \cosh(t)\, dt$$

$$= \frac{1}{2}\int \sinh(2t)\,dt$$
$$= \frac{1}{4}\cosh(2t) + c$$
$$= \frac{1}{4}\cosh(2\arg\cosh(x+2)) + c,\ c \in \mathbb{R}$$

-b) We have

$$J = \int \frac{x}{\sqrt{x^2+4x+5}}dx$$
$$= \int \frac{x}{\sqrt{(x+2)^2+1}}dx.$$

We make the subtitution $x+2 = sh(t)$, $\quad t \in \mathbb{R}$, we obtain

$$J = \int \frac{x}{\sqrt{(x+2)^2+1}}dx$$

$$= \int \frac{-2+\sinh(t)}{\cosh(t)} ch(t)\,dt$$
$$= \int (-2+\sinh(t))\,dt$$
$$= -2t + \cosh(t) + c$$
$$= -2\arg\sinh(x+2) + \sqrt{x^2+4x+5} + c,\ c \in \mathbb{R}.$$

9.4 Primitives of composite trigonometric functions

9.4.1 Product of sine or cosine

9.4.1.1 Integral of $f(x) = (\sin(x))^p(\cos(x))^q$, $p,\ q \in \mathbb{N}$

Case when p or q is a positive odd number : We will calculate $\int (\sin(x))^p(\cos(x))^{2k+1}dx$.

Suppose that $q = 2k+1$ (q to be o dd and positive) and p an arbitrary natural number. We write

$$(\sin(x))^p(\cos(x))^{2k+1} = (\sin(x))^p(\sin(x))^{2k}\cos(x)$$
$$= (\sin(x))^p\left(1-\sin^2(x)\right)^k\cos(x),$$

We put $t = \sin(x)$, we have

$$\int t^p\left(1-t^2\right)^k dt,$$

Example 45. We will calculate $I = \int (\sin(x))^3 (\cos(x))^2\, dx$. The first factor can be expanded, and the terms integrated separately. We write

$$I = \int (\sin(x))^3 (\cos(x))^2\, dx = \int \left(1 - \cos^2(x)\right) \cos^2(x) \sin(x)\, dx.$$

We put $t = \cos(x)$, we have

$$\begin{aligned} I = \int t^2 \left(1 - t^2\right) dt &= \frac{1}{5} t^5 - \frac{1}{3} t^3 + c, \\ &= \frac{1}{5} \cos^5(x) - \frac{1}{3} \cos^3(x) + c, \quad c \in \mathbb{R}. \end{aligned}$$

Case when p and q even : We will calculate $\int (\sin(x))^{2h} (\cos(x))^{2k}\, dx,\ h,\ k \in \mathbb{N}$.

a) If p and q are not largs, we use the trigonometric transformation of $(\sin(x))^{2h}$ and $(\cos(x))^{2k}$:

Example 46. We will calculate $I = \int (\sin(x))^4 (\cos(x))^2\, dx$. We write

$$\begin{aligned} (\sin(x))^4 (\cos(x))^2 &= \frac{1}{4} \sin^2(x) \sin^2(2x) \\ &= \frac{1}{4} \times \frac{1 - \cos(2x)}{2} \times \frac{1 - \cos(4x)}{2} \\ &= \frac{1}{16} (1 - \cos(2x) - \cos(4x) + \cos(2x)\cos(4x)) \\ &= \frac{1}{16} \left(1 - \cos(2x) - \cos(4x) + \frac{1}{2}\cos(2x) + \frac{1}{2}\cos(2x)\right). \end{aligned}$$

Then

$$I = \int (\sin(x))^4 (\cos(x))^2\, dx$$

.

$$\begin{aligned} &= \frac{1}{16} \int (1 - \cos(2x) - \cos(4x) + \cos(2x)\cos(4x))\, dx \\ &= \frac{1}{16} \left(x - \frac{1}{4}\sin(2x) - \frac{1}{4}\sin(4x) + \frac{1}{12}\sin(6x)\right). \end{aligned}$$

b) Theoretical, to calculate $\int (\sin(x))^{2p} (\cos(x))^{2q}\, dx,\ p,\ q \in \mathbb{N}$, we use the integration by parts and by indiction (by successive reduction) we obtain the result.

Second method: Linearity

Example 47. Calculons $I = \int \cos^4(x)\, dx$.

We have

$$2\cos(x) = e^{ix} + e^{-ix},$$

we devolope, we obtain

$$\begin{aligned} 2^4 \cos^4(x) &= e^{4ix} + 4e^{2ix} + 6 + 4e^{-2ix} + e^{-4ix} \\ &= e^{4ix} + e^{-4ix} + 4\left(e^{2ix} + e^{-2ix}\right) + 6 \\ &= 2\cos(4x) + 8\cos(2x) + 6. \end{aligned}$$

Therefore

$$\begin{aligned} I &= \int \cos^4(x)\, dx \\ &= \frac{1}{8}\int \cos(4x)\, dx + \frac{1}{2}\int \cos(2x)\, dx + \frac{3}{8}\int dx. \\ &= \frac{1}{32}\sin(4x) + \frac{1}{4}\sin(2x) + \frac{3}{8}x + c, \quad c \in \mathbb{R}. \end{aligned}$$

9.4.2 Rational fraction of $\sin(x)$ and $\cos(x)$

We will calculate

$$I = \int f(\cos(x), \sin(x))\, dx.$$

Put

$$t = \tan\left(\frac{x}{2}\right),\ x \in \left]-\pi, \pi\right[.$$

So $x = 2\arctan(t)$ is of a class $C^1(\mathbb{R}, \left]-\pi, \pi\right[)$, we have

$$dx = \frac{2}{1+t^2}dt.$$

Therefore, we comme back to the calculate of primitive for rational fraction.

$$I = \int f\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\frac{1}{1+t^2}dt.$$

Example 48. We will calculate

$$I = \int \frac{1}{\cos(x)}dx.$$

Put

$$t = \tan\left(\frac{x}{2}\right),\ x \in \left]-\pi, \pi\right[,$$

then

$$dx = \frac{2}{1+t^2}dt.$$

So,

$$\begin{aligned} I &= 2\int \frac{1+t^2}{1-t^2}\frac{1}{1+t^2}dt \\ &= \int \frac{1}{1-t^2}dt \\ &= \ln\left|\frac{1+t}{1-t}\right| + c \\ &= \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| + c,\ c \in \mathbb{R}. \end{aligned}$$

9.5 Itegration of hyperbolic functions

We will determinate the primitive of rational fraction of the form $F(\sinh(x), \cosh(x), \tanh(x))$.

For calculate the primitive of

$$F(x) = \int f(\sinh(x), \cosh(x))\,dx,$$

we use the change of variable

$$t = \tanh\left(\frac{x}{2}\right), \quad x \in \mathbb{R},$$

with

$$\cosh(x) = \frac{1+t^2}{1-t^2},\ \sinh(x) = \frac{2t}{1-t^2},\ \textit{and}\ dx = \frac{2}{1-t^2}dt.$$

Therefore, we comme back to the calculate of primitive for rational fraction

$$I = \int f\left(\frac{2t}{1-t^2}, \frac{1+t^2}{1-t^2}\right)\frac{2}{1-t^2}dt.$$

Remark 31. 1) We can make the substitution (change de variable) $t = e^x$.

2) When the subtitution rule (change of variable) not allows to come back to the integral of rational fractions, we use integration by part to obtain the integral.

Example 49. We will calculate

$$I = \int \frac{dx}{\sinh(x)}.$$

We make the substitution (change of the variable)

$$t = \tanh\left(\frac{x}{2}\right), \quad x \in \mathbb{R},$$

Consequently, we have

$$\begin{aligned} I &= \int \frac{dx}{\sinh(x)} \\ &= \int \frac{1}{\frac{2t}{1-t^2}} \cdot \frac{2}{1-t^2}dt \end{aligned}$$

$$= \int \frac{1}{t} dt$$

$$= ln\,|t| + c = \ln \left| \tanh \left(\frac{x}{2} \right) \right| + c, \ where \, c \in \mathbb{R}.$$

Example 50. We will calculate

$$I = \int \frac{dx}{ch\,(x)}.$$

we make the substitution (change of the variable)

$$t = \tanh \left(\frac{x}{2} \right), \quad x \in \mathbb{R},$$

Consequently, we have

$$I = \int \frac{dx}{\cosh (x)}$$

$$= \int \frac{1}{\frac{1+t^2}{1-t^2}} \cdot \frac{2}{1-t^2} dt$$

$$= \int \frac{1}{1+t^2} dt$$

$$= \arctan (t) + c = \arctan \left[\tanh \left(\frac{x}{2} \right) \right] + c, \ where \, c \in \mathbb{R}.$$

Example 51. We calculate

$$\int \frac{dx}{\cosh (3x) - \cosh (x)}.$$

First, by using hyperbolic formulas, we have

$$\begin{aligned}
\cosh (3x) - \cosh (x) &= \cosh (2x + x) - \cosh (x) \\
&= \cosh (2x) \cosh (x) + \sinh (2x) \sinh (x) - \cosh (x) \\
&= (\cosh (2x) - 1) \cosh (x) + \sinh (2x) \sinh (x) \\
&= \left(2 \sinh^2 (x) \right) \cosh (x) + 2 \sinh (x) \cosh (x) \sinh (x) \\
&= 2 \sinh^2 (x) \cosh (x) + 2 \sinh^2 (x) \cosh (x) \\
&= 4 \sinh^2 (x) \cosh (x).
\end{aligned}$$

Then, we make the substitution (change the variable) $t = \sinh (x)$, we obtain

$$\begin{aligned}
\int \frac{dx}{\cosh (3x) - \cosh (x)} &= \int \frac{dt}{4t^2 (t^2 + 1)} \\
&= \frac{1}{4} \int \left(\frac{1}{t^2} - \frac{1}{t^2 + 1} \right) dt \\
&= -\frac{1}{4t} - \frac{1}{4} \arctan (t) \\
&= -\frac{1}{4 \sinh (x)} - \frac{1}{4} \arctan (\sinh (x)).
\end{aligned}$$

9.6 Primitives of irrational functions

We will calculate
$$I=\int f\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx.$$
By substitution rule, we have
$$y=\sqrt[n]{\frac{ax+b}{cx+d}}, \quad ad-bc \neq 0.$$
Then we have the inverse function from y to x,
$$x=\frac{b-dy^n}{cy^n-a}.$$
Thus
$$I=\int f\left(\frac{b-dy^n}{cy^n-a}, y\right) \frac{ad-bc}{(cy^n-a)^2} ny^{n-1} dy.$$
This yields to calculate the primitive of rational function.

Example 52. We calculate
$$I=\int \sqrt{\frac{1+x}{1-x}} \cdot \frac{1}{2x-1} dx.$$
Put
$$y=\sqrt{\frac{1+x}{1-x}}, \ \ x \in [-1,1[\,,$$
then
$$x=\frac{y^2-1}{y^2+1}, \quad y \in \mathbb{R}.$$
By the substitution rule (put $x=\frac{y^2-1}{y^2+1}, \quad y \in \mathbb{R}$) gives
$$\begin{aligned} I &= \int \frac{4y^2}{(y^2-3)(y^2+1)} dy \\ &= \int \frac{3}{y^2-3} dy + \int \frac{1}{y^2+1} dy \\ &= \frac{\sqrt{3}}{2} \ln \left| \frac{y-\sqrt{3}}{y+\sqrt{3}} \right| + c, \quad c \in \mathbb{R}. \end{aligned}$$
Last we remplace y by $\sqrt{\frac{1+x}{1-x}}$ to obtained the primitive function.

2) Integration by rationalization and integration by substitution a) Some irrational expressions may be integrated by substituting a new variable, so related to the old, that the new expression shall be rational.

b) Expressions involving only fractional powers of x. Such forms may be rationalized by assuming $x = z^n$ where n is the least common multiple of the denominators of the several fractional exponents.

Example 53. We will calculate
$$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx.$$
Put $z = x^6 \Rightarrow dz = 6x^5 dx$. Then
$$x^{\frac{1}{2}} = z^3, \quad x^{\frac{1}{3}} = z^2.$$

Hence
$$\begin{aligned} \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx &= \int \frac{6z^5}{z^3 + z^2} dz \\ &= \int \frac{6z^3}{z+1} dz \\ &= \int \left(z^2 - z + 1\right) dz - \int \frac{1}{z+1} dz \\ &= \frac{1}{3} z^3 - \frac{1}{2} z^2 + z - \ln|z+1| + c. \end{aligned}$$
Substituting in this $z = x^{\frac{1}{6}}$, we have
$$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx = 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6 \ln\left|x^{\frac{1}{6}} + 1\right| + c.$$

Exercise 48. Calculate the following indefinit integrals:

1) $\int \frac{5x^2+4x+3}{x(x-1)(x+3)} dx$
2) $\int \frac{\sqrt{1-\sqrt{x}}}{x}$, $x \in]0, 1[$.
3) $\int \cos^3(x) \sin^6(x)\, dx$
4) $\int \cos^8(x) \sin^5(x)\, dx$
5) $\int \cos^2(x) \sin^4(x)\, dx$
6) $\int \tan^3(x)\, dx$
7) $\int \frac{x^2}{1+x^6} dx$
8) $\int x e^{-x^2} \sin\left(e^{-x^2}\right) dx$
9) $\int \frac{dx}{\sinh(x) \cosh^2(x)}$
10) $\int \frac{\sinh(x)}{\sqrt{\cosh(2x)}} dx$
10) $\int \frac{dx}{2\sinh(x)+3\cosh(x)}$

Solution: We decompose the frantion to simple elements in $\mathbb{R}[X]$, we obtain

$$\frac{5x^2+4x+3}{x(x-1)(x+3)} = -\frac{1}{x} + \frac{3}{x-1} + \frac{3}{x+3},$$

then

$$\begin{aligned}\int \frac{5x^2+4x+3}{x(x-1)(x+3)} &= -\int \frac{1}{x}dx + \int \frac{3}{x-1}dx + \int \frac{3}{x+3}dx \\ &= -\ln|x| + \ln|x-1| + \ln|x+3| + c, \quad c \in \mathbb{R}.\end{aligned}$$

2) To calculate $\int \frac{\sqrt{1-\sqrt{x}}}{x}$, $x \in \left]0,1\right[$, we make the substitution (change the variable) $u = \sqrt{1-\sqrt{x}}$, we obtain

$$\begin{cases} u^2 = 1-\sqrt{x}, \\ x = (1-u^2)\,u^2. \end{cases}$$

The function $x(u) = (1-u^2)\,u^2$ is a bijection increasing from $\left]0,1\right[$ to $\left]0,1\right[$ and $u(x)$ and $x(u)$ are of class C^∞. Then $dx = -4(1-u^2)\,u$. Therefore

$$\begin{aligned}\int \frac{u}{1-u^2}\left[-4\left(1-u^2\right)u\right]du &= \int \frac{-4u^2}{1-u^2}du \\ &= \int \left(4 - \frac{4}{1-u^2}\right)du \\ &= \int \left(4 + \frac{2}{1-u} - \frac{2}{1+u}\right)du \\ &= 4u + 2\left(\ln(u-1) - \ln(u+1)\right) + c,\ c \in \mathbb{R} \\ &= 4u + 2\ln\left(\frac{u-1}{u+1}\right) + c,\ c \in \mathbb{R}.\end{aligned}$$

Thus

$$\int \frac{\sqrt{1-\sqrt{x}}}{x} = 4\sqrt{1-\sqrt{x}} + 2\ln\left(\frac{\sqrt{1-\sqrt{x}}-1}{\sqrt{1-\sqrt{x}}+1}\right) + c,\ x \in \left]0,1\right[.$$

3) To calculate $\int \cos^3(x)\sin^6(x)\,dx$, we make the substitution (change the variable) $u = \sin(x) \Longrightarrow du = \cos(x)\,dx$, we obtain

$$\begin{aligned}\int \cos^3(x)\sin^6(x)\,dx &= \int \cos^2(x)\sin^6(x)\cos(x)\,dx \\ &= \int \left(1-\sin^2(x)\right)\sin^6(x)\cos(x)\,dx \\ &= \int \left(1-u^2\right)u^6du \\ &= \int \left(u^6-u^8\right)du\end{aligned}$$

$$= \frac{u^7}{7} - \frac{u^9}{9} + c$$
$$= \frac{\sin^7(x)}{7} - \frac{\sin^9(x)}{9} + c,\ c \in \mathbb{R}.$$

4) To calculate $\int \cos^8(x)\sin^5(x)\,dx$, we make the substitution (change the variable) $u = \cos(x) \Longrightarrow du = -\sin(x)\,dx$, on obtain

$$\begin{aligned}
\int \cos^8(x)\sin^5(x)\,dx &= -\int \cos^8(x)\sin^4(x)\sin(x)\,dx \\
&= -\int \left(1-\cos^2(x)\right)^2 \cos^8(x)\sin(x)\,dx \\
&= -\int \left(1-u^2\right)^2 u^8 du \\
&= -\int \left(u^8 - 2u^{10} + u^{12}\right) du \\
&= -\frac{u^9}{9} + 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + c,\ c \in \mathbb{R} \\
&= -\frac{\cos^9(x)}{9} + 2\frac{\cos^{11}(x)}{11} + \frac{\cos^{13}(x)}{13} + c.
\end{aligned}$$

5) $\int \cos^2(x)\sin^4(x)\,dx$. We use the linear expression, we have

$$\begin{aligned}
\cos^2(x)\sin^4(x) &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^2 \left(\frac{e^{ix}-e^{-ix}}{2}\right)^4 \\
&= \frac{1}{32}\left(\frac{e^{6ix}+e^{-6ix}}{2}\right)^2 - 2\frac{e^{4ix}+e^{-4ix}}{2} + \frac{e^{2ix}+e^{-2ix}}{2} + 2 \\
&= \frac{1}{16} + \frac{\cos(6x)}{32} - \frac{\cos(4x)}{16} - \frac{\cos(2x)}{32}
\end{aligned}$$

Then

$$\begin{aligned}
\int \cos^2(x)\sin^4(x)\,dx &= \int \left[\frac{1}{16} + \frac{\cos(6x)}{32} - \frac{\cos(4x)}{16} - \frac{\cos(2x)}{32}\right] dx \\
&= \frac{1}{16}x + \frac{\sin(6x)}{192} - \frac{\sin(4x)}{64} - \frac{\sin(2x)}{64} + c, \quad c \in \mathbb{R}.
\end{aligned}$$

6)

$$\begin{aligned}
\int \tan^3(x)\,dx &= \int \tan^2(x)\tan(x)\,dx \\
&= \int \left(-1+1+\tan^2(x)\right)\tan(x)\,dx \\
&= \int \left(1+\tan^2(x)\right)\tan(x)\,dx - \int \tan(x)\,dx
\end{aligned}$$

$$= \frac{1}{2}\tan^2(x) - \int \frac{\sin(x)}{\cos(x)}dx$$
$$= \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + c, \quad where\ c \in \mathbb{R}.$$

7)
$$\int \frac{x^2}{1+x^6}dx = \int \frac{x^2}{1+(x^3)^2}dx$$
$$= \frac{1}{3}\int \frac{3x^2}{1+(x^3)^2}dx$$
$$= \frac{1}{3}\arctan\left(x^3\right) + c, \quad where\ c \in \mathbb{R}.$$

8)
$$\int xe^{-x^2}\sin\left(e^{-x^2}\right)dx = \frac{1}{2}\int 2xe^{-x^2}\sin\left(e^{-x^2}\right)dx$$
$$= \frac{1}{2}\cos\left(e^{-x^2}\right) + c, \quad where\ c \in \mathbb{R}.$$

9)
$$\int \frac{dx}{\sinh(x)\cosh^2(x)} = \int \frac{\cosh^2(x) - \sinh^2(x)}{\sinh(x)\cosh^2(x)}dx$$
$$= \int \frac{dx}{\sinh(x)} - \int \frac{\sinh(x)}{\cosh^2(x)}dx$$
$$= \int \frac{dx}{\sinh(x)} - \int \frac{d(\cosh(x))}{\cosh^2(x)}$$
$$= \ln\left|\tan\left(\frac{x}{2}\right)\right| + \frac{1}{\cosh(x)} + c,\ where\ c \in \mathbb{R}.$$

10)
$$\int \frac{\sinh(x)}{\sqrt{\cosh(2x)}}dx = \int \frac{\sinh(x)}{\sqrt{2\cosh^2(x) - 1}}dx$$
$$= \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}\sinh(x)}{\sqrt{\left(\sqrt{2}\cosh(x)\right)^2 - 1}}dx$$
$$= \frac{1}{\sqrt{2}}\int \frac{d\left(\sqrt{2}\sinh(x)\right)}{\sqrt{\left(\sqrt{2}\cosh(x)\right)^2 - 1}}$$
$$= \frac{1}{\sqrt{2}}\ln\left|\sqrt{2}\cosh(x) + \sqrt{\left(\sqrt{2}\cosh(x)\right)^2 - 1}\right| + c, wherec \in \mathbb{R}.$$

10) By using the exponential expression, we obtain

$$\int \frac{dx}{2\sinh(x) + 3\cosh(x)} = \int \frac{dx}{5e^x + e^{-x}}$$

$$= 2\int \frac{dx}{5e^{2x}+1}$$
$$= \frac{2}{\sqrt{5}}\int \frac{d\left(\sqrt{5}e^{x}\right)}{\left(\sqrt{5}e^{x}\right)^{2}+1}$$
$$= \frac{2}{\sqrt{5}}\arctan\left(\sqrt{5}e^{x}\right)+c,\ where\ c\in\mathbb{R}.$$

Exercise 49. Calculate the following integrals:

1) $\int_{-1}^{1}\frac{2x-5}{x^2-5x+6}$.
2) $\int_0^{\frac{\pi}{3}}\cos^3(x)\,dx$.

Solution: 1)

$$\int_{-1}^{1}\frac{2x-5}{x^2-5x+6} = \left[\ln\left|x^2-5x+6\right|\right]_{-1}^{1} = -\ln(2)-\ln(3) = -\ln(6).$$

.

2)

$$\int_0^{\frac{\pi}{3}}\cos^3(x)\,dx = \int_0^{\frac{\pi}{3}}\cos^2(x)\cos(x)\,dx$$
$$= \int_0^{\frac{\pi}{3}}\left(1-\sin^2(x)\right)\cos(x)\,dx$$
$$= \int_0^{\frac{\pi}{3}}\cos(x)\,dx - \int_0^{\frac{\pi}{3}}\sin^2(x)\cos(x)\,dx$$
$$= [\sin(x)]_0^{\frac{\pi}{3}} - \left[\frac{\sin^3(x)}{3}\right]_0^{\frac{\pi}{3}} = \frac{9\sqrt{3}-\pi}{24}.$$

Integration by part. Integration by successive reduction :

Exercise 50. For all $n\in\mathbb{N}$, we set (put)

$$I_n = \int_0^1\left(1-x^2\right)^n dx.$$

1) Find the relation between I_n and I_{n-1} for all $n\in\mathbb{N}^n$.
2) Deduce the value of I_n .
3) Deduce the value of $\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}C_n^k$.

Solution: 1) By integration by part, we obtain

$$\begin{aligned}
I_n &= \int_0^1 \left(1-x^2\right)^n dx \\
&= \left[x\left(1-x^2\right)^n\right]_0^1 + 2n\int_0^1 x^2\left(1-x^2\right)^{n-1} dx \\
&= 2n\int_0^1 \left(x^2-1+1\right)\left(1-x^2\right)^{n-1} dx \\
&= 2n\int_0^1 \left(x^2-1\right)\left(1-x^2\right)^{n-1} dx + 2n\int_0^1 \left(1-x^2\right)^{n-1} dx \\
&= -2n\int_0^1 \left(1-x^2\right)^n dx + 2n\int_0^1 \left(1-x^2\right)^{n-1} dx \\
&= -2nI_n + 2nI_{n-1}.
\end{aligned}$$

Thus

$$I_n = \frac{2n}{2n+1} I_{n-1}.$$

2) We have

$$\begin{aligned}
I_n &= \frac{2n}{2n+1} I_{n-1} = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1} I_{n-2} \\
&= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3} I_{n-3} \\
&= \cdots = \frac{2n\,(2n-2)\,(2n-4)\cdots 2}{(2n+1)\,(2n-1)\,(2n-3)\cdots 3} I_0 \\
&= \frac{(2n)^2\,(2n-2)^2\,(2n-4)^2\cdots 2^2}{(2n+1)\,(2n)\,(2n-1)\,(2n-2)\,(2n-3)\,(2n-4)\cdots 3\cdot 2} I_0 \\
&= \frac{\left[2n\,(2n-2)\,(2n-4)\cdots 1\right]^2}{(2n+1)!} I_0 \\
&= \frac{\left[2^n n\,(n-1)\,(n-2)\cdots 2\right]^2}{(2n+1)!} I_0 \\
&= \frac{\left[2^n n!\right]^2}{(2n+1)!} I_0 \\
&= \frac{\left[2^n n!\right]^2}{(2n+1)!},
\end{aligned}$$

because $I_0 = 1$.
3) We have

$$
\begin{aligned}
I_n &= \int_0^1 \left(1-x^2\right)^n dx \\
&= \int_0^1 \sum_{k=0}^{n} C_n^k \,(1)^{n-k} \left(-x^2\right)^k dx \\
&= \int_0^1 \sum_{k=0}^{n} C_n^k \left(-x^2\right)^k dx \\
&= \int_0^1 \sum_{k=0}^{n} C_n^k \,(-1)^k \left(-x^2\right)^{2k} dx \\
&= \sum_{k=0}^{n} (-1)^k \, C_n^k \left[\frac{x^{2k+1}}{2k+1}\right]_0^1 \\
&= \sum_{k=0}^{n} (-1)^k \, C_n^k \left[\frac{1}{2k+1}\right]
\end{aligned}
$$

Thus

$$\sum_{k=0}^{n} C_n^k \left(\frac{(-1)^k}{2k+1}\right) = \frac{\left[2^n n!\right]^2}{(2n+1)!}.$$

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Bibliography

[1] K. Allab, Eléments d'analyse, Fonctions d'une variable réelle, Entreprise Nationale du Livre. Alger. 1990.

[2] A. Benyad et M. Benhassaine, Mathématique, Exercices résolus, O. P. U. Cinquième Edition, 2014.

[3] C. Baba-Hamed et K. Benhabib, Rappels de cours et exercices avec solutions, O. P. U. Alger 1998.

[4] A. Doneddu, Topologie, Fonctions réelles d'une variable réelle, Tome 4, Deuxième Edition, Librairie Vuibert, 1979.

[5] B. P. Démidovitch, Recueil d'exercices et de problèmes d'analyse mathématique, Edition Naouka (en Russe), 1990.

[6] M. Mehbali, Mathématiques 1, Fonction d'une variable réelle, O. P. U. Alger. Edition, 2009.

[7] Paul Seeburger, MTH 211 Calculus II, Monroe Community College, 2023, LibreText (https://LibreTexts.org) , info@LibreTexts.org, (https://facebook.com/Libretexts), Twitter (https://twitter.com/libretexts), or our blog (http://Blog.Libretexts.org).

[8] W. Trench, Introduction to real analysis, Department of Mathematics Trinity University San Antonio, Texas, USA, 2002, wtrench@trinity.edu.

[9] Gilbert Strang, Calculus, Massachussete Institute of Technology, Wellesly Combridg Press. Box 82-279, Wellesly MA 02181.