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Pedagogical handout

Courses and application exercises intended for second-year preparatory students

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Introduction

This handout is primarily intended for second-year preparatory students at the Higher School of Economics. It has been designed to deepen your understanding of fundamental concepts in mathematical analysis through the study of numerical series, improper integrals, ordinary differential equations, and functions of two variables.

Each section of this handout begins with an exploration of essential theoretical foundations, followed by detailed theoretical propositions and a series of diverse practical exercises. This pedagogical format aims to help you develop your analytical skills and strengthen your theoretical knowledge, thereby preparing you to tackle more advanced mathematical challenges that will be necessary in your future studies in economics.

The purpose of this handout is to acquire a thorough and rigorous understanding of the methods and fundamental concepts of mathematical analysis, essential for effective application in various fields of economics. We hope that this material will significantly enrich your learning and help you excel in your studies.

Before concluding, we would like to remind you, dear students, that the key to solving exercises and problems lies in understanding the theory. As the proverb says, "Give a man a fish and you feed him for a day; teach a man to fish and you feed him for a lifetime."

Chapter 1

Numerical Series

1.1 Fundamentals of numerical series

Definition 1.1.1. *Let* (a_n) *be a sequence of elements in* $\mathbb{R}$*. The expression* $a_0 + a_1 + \cdots + a_n + \cdots = \sum_{n\geqslant 0} a_n$ *is called a numerical series.*

The terms $a_0, a_1, a_2, \ldots, a_n$ *are called terms of the series.*

The term a_n *is called the general term of the series.*

Notation

- $S_1 = a_1$.
- $S_2 = a_1 + a_2$.
- $\ldots$
- $S_n = a_1 + a_2 + \cdots + a_n$.

The sums $S_1, \ldots, S_n$ are called partial sums of the series, and $(S_n)_{n\in\mathbb{N}}$ is the sequence of partial sums of the series.

Definition 1.1.2. *We say that the series* $\sum_{n\geqslant 0} a_n$ *converges if the sequence* $(S_n)_{n\in\mathbb{N}}$ *has a finite limit* $S \in \mathbb{R}$*, and we write* $\displaystyle\sum_{n\geqslant 0} a_n = S$.

Example 1.1.3. $\displaystyle\sum_{n\geqslant 1} \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

- $a_1 = 1 - \frac{1}{2}$
- $a_2 = \frac{1}{2} - \frac{1}{3}$
- $\ldots$
- $a_{n-1} = \frac{1}{n-1} - \frac{1}{n}$
- $a_n = \frac{1}{n} - \frac{1}{n+1}$

Thus, by summation, we obtain $S_n = 1 - \frac{1}{n+1} \to 1$ *as* $n \to +\infty$. *Hence, the series converges and* $\displaystyle\sum_{n \geqslant 1} a_n = 1$.

Remark 1.1.4.. —

The series $\displaystyle\sum_{n \geqslant 0} a_n$ and $\displaystyle\sum_{n \geqslant k} a_n$ with $k \in \mathbb{N}$ are of the same nature; in other words, the convergence of a series does not depend on its initial terms.

Definition 1.1.5. *For* $\displaystyle\sum_{n \geqslant 0} a_n$ *a convergent series, the expression* $\displaystyle\sum_{k \geqslant n+1} a_k$ *is called the remainder of the series.*

Proposition 1.1.6. *For a convergent series* $\displaystyle\sum_{n \geqslant 0} a_n$, *the remainder* $R_n = \displaystyle\sum_{k \geqslant n+1} a_k$ *tends to* 0 *as* $n \to +\infty$.

Proof — Indeed, if $\displaystyle\sum_{n \geqslant 0} a_n$ converges, then its partial sum $S_n = \displaystyle\sum_{k=0}^{n} a_k$ converges to $S \in \mathbb{R}$, and we have $S = S_n + R_n$, thus $R_n = S - S_n$ tends to 0.

Proposition 1.1.7. *For a convergent series* $\displaystyle\sum_{n \geqslant 0} a_n$, *we have* $\displaystyle\lim_{n \to +\infty} a_n = 0$.

Proof — *For all* $n \geqslant 0$, *let* $S_n = \displaystyle\sum_{k=0}^{n} a_k$.

For all $n \geqslant 1$, $a_n = S_n - S_{n-1}$. *If the series* $\displaystyle\sum_{n \geqslant 0} a_n$ *converges, then its partial sum* S_n *converges to* S, *and the partial sum* S_{n-1} *also converges. Taking the limit, we get* $\displaystyle\lim_{n \to +\infty} a_n = S - S = 0$.

Remark 1.1.8.. —

The converse is false.

Counter-example.

Consider the series $\sum_{n\geqslant 1} \log\left(1+\frac{1}{n}\right)$. *It's evident that the general term of this series converges to* 0. *Now, let's calculate the partial sum:*

$$\log\left(1+\frac{1}{n}\right) = \log(n+1) - \log(n)$$

The terms of the series can be written as:

- $a_1 = \log(2) - \log(1)$
- $a_2 = \log(3) - \log(2)$
- $\ldots$
- $a_{n-1} = \log(n) - \log(n-1)$
- $a_n = \log(n+1) - \log(n)$

By summation, we have $S_n = \log(n+1)$ *which tends to* $+\infty$ *as* $n \to +\infty$.

Remark 1.1.9.. —

The previous proposition is often used in its contrapositive form. For instance, the series $\sum_{n\geqslant 1} \ln(n)$ *is divergent because its general term tends to* $+\infty$, *which is not* 0.

1.1.1 Geometric Series

Definition 1.1.10. *The series* $\sum_{n\geqslant 0} k^n$ *with* $k \in \mathbb{R}$ *is called a geometric series.*

Proposition 1.1.11. *The geometric series* $\sum_{n\geqslant 0} k^n$ *converges if and only if* $-1 < k < 1$.

PROOF — *Indeed, we can easily calculate the partial sum:*

- *If* $k \neq 1$, *then:*

  $$S_n = 1 + k + k^2 + \ldots + k^n = \frac{1-k^{n+1}}{1-k}$$

  This series converges if and only if $-1 < k < 1$, *and its limit is* $\lim_{n\to+\infty} S_n = \frac{1}{1-k}$.

- *If* $k = 1$, *then* $S_n = 1 + 1 + \ldots + 1 = n + 1$ *and* $\lim_{n\to+\infty} S_n = +\infty$.

1.1.2 Riemann Series

Definition 1.1.12. *The series* $\sum_{n\geqslant 1}\frac{1}{n^p}$ *is called the Riemann series.*

Proposition 1.1.13. *The Riemann series converges if and only if* $p>1$.

Proof — *The proof will be provided on page 15 .*

Remark 1.1.14.. —

In the case where $p=1$, *the resulting series is called the harmonic series.*

Proposition 1.1.15.

1) *Let* $\sum_{n\geqslant 0} a_n$ *and* $\sum_{n\geqslant 0} b_n$ *be two convergent numerical series. Then* $\sum a_n+b_n$ *is convergent and we have:*

$$\sum a_n+b_n=\sum_{n\geqslant 0} a_n+\sum_{n\geqslant 0} b_n$$

2) *If* $\sum_{n\geqslant 0} a_n$ *converges, then for* $c\in\mathbb{R}$, $\sum_{n\geqslant 0} ca_n$ *converges and we have* $\sum_{n\geqslant 0} ca_n=c\sum_{n\geqslant 0} a_n$.

It is not always possible to calculate the partial sum; for this, convergence criteria are used.

1.2 Series with Nonnegative Terms:

Definition 1.2.1. *A numerical series* $\sum_{n\geqslant 0} a_n$ *is said to have nonnegative terms if for all* $n\geqslant 0$, $a_n\geqslant 0$.

1.2.1 Necessaries and Sufficients Conditions for Convergence

Proposition 1.2.2. *A series* $\sum_{n\geqslant 0} a_n$ *with nonnegative terms is convergent if and only if the sequence of partial sums* S_n *is bounded.*

Proof — *Since* $a_n \geqslant 0$*, the sequence* (S_n) *is necessarily increasing.*
Assume $\sum_{n\geqslant 0} a_n$ *converges, then* $S_n \to S \in \mathbb{R}$*, hence* (S_n) *is bounded.*
Assume now that the sequence $(S_n)_{n\in\mathbb{N}}$ *is bounded, i.e.,*

$$\exists M > 0 \quad \forall n \in I\!N, \quad S_n < M$$

Since (S_n) *is increasing and bounded, it converges to a limit* $S \in \mathbb{R}$*, thus* $\sum_{n\geqslant 0} a_n$ *converges.*

1.2.2 Convergence Criteria for Series with Nonnegative Terms

Theorem 1.2.3.. — ***(First Version of the Comparison Criterion)***

Let $\sum_{n\geqslant 0} a_n$ *and* $\sum_{n\geqslant 0} b_n$ *be two series with nonnegative terms, and let* $N \in \mathbb{N}$*. Then:*

1) *If* $a_n \leqslant b_n$ *for all* $n \geqslant N$ *and* $\sum_{n\geqslant 0} b_n$ *converges, then* $\sum_{n\geqslant 0} a_n$ *converges.*

2) *If* $b_n \leqslant a_n$ *for all* $n \geqslant N$ *and* $\sum_{n\geqslant 0} b_n$ *diverges, then* $\sum_{n\geqslant 0} a_n$ *diverges.*

Proof —

1) *Let* $(S_n)_n$ *and* $(T_n)_n$ *be the sequences of partial sums for* $\sum_{n\geqslant N} a_n$ *and* $\sum_{n\geqslant N} b_n$ *respectively. Suppose* $a_n \leqslant b_n$ *for all* $n \geqslant N$*, then* $S_n \leqslant T_n$ *for all* $n \geqslant N$*. If* $\sum_{n\geqslant 0} a_n$ *converges, by the previous proposition,* $(T_n)_n$ *is bounded, and since* $S_n \leqslant T_n$*, it follows that* $(S_n)_n$ *is also bounded. Thus,* $\sum_{n\geqslant 0} a_n$ *converges.*

2) *If* $\sum_{n\geqslant 0} b_n$ *diverges, then the sequence* $T_n \to +\infty$*, hence* $S_n \to +\infty$*, showing that* $\sum_{n\geqslant 0} a_n$ *diverges.*

Example 1.2.4. *We want to study the convergence of the series* $\sum_{n\geqslant 1} \frac{1}{n^2+1}$*. By comparison with the Riemann series* $\sum_{n\geqslant 1} \frac{1}{n^2}$*, for all* $n \geqslant 1$*:*

$$\frac{1}{n^2+1} < \frac{1}{n^2}$$

and since the series $\sum_{n\geqslant 1}\frac{1}{n^2}$ *converges,* $\sum_{n\geqslant 1}\frac{1}{n^2+1}$ *also converges.*

Theorem 1.2.5.. — ***(Second Version of the Comparison Criterion)***

Let $\sum_{n\geqslant 0} a_n$ *and* $\sum_{n\geqslant 0} b_n$ *be two series with nonegatives terms.*
If there exist constants $A > 0$*,* $B > 0$*, and a natural number* N *such that*

$$\forall n \geqslant N, \quad A \leqslant \frac{a_n}{b_n} \leqslant B,$$

then the two series have the same nature. Proof — *The result follows from* $Ab_n \leqslant a_n \leqslant Bb_n$ *for all* $n \in \mathbb{N}$*.*
Indeed, if $\sum_{n\geqslant 0} a_n$ *converges, then by the previous theorem,* $\sum_{n\geqslant 0} Ab_n$ *converges, hence* $\sum_{n\geqslant 0} b_n$ *converges.*
Assume now $\sum_{n\geqslant 0} b_n$ *converges, then* $\sum_{n\geqslant 0} Bb_n$ *also converges, thus by the previous theorem,* $\sum_{n\geqslant 0} a_n$ *converges. Therefore, the series have the same nature.*

Corollary 1.2.1. *Let* $\sum_{k\geqslant 0} a_n$ *and* $\sum_{n\geqslant 0} b_n$ *be two series with positive terms.*

- *If* $\lim_{n\to+\infty}\frac{a_n}{b_n} = c > 0$ *then the series have the same nature.*
- *If* $\lim_{n\to+\infty}\frac{a_n}{b_n} = 0$ *and* $\sum_{n\geqslant 0} b_n$ *converges then* $\sum_{n\geqslant 0} a_n$ *converges.*
- *If* $\lim_{n\to+\infty}\frac{a_n}{b_n} = +\infty$ *and* $\sum_{n\geqslant 0} b_n$ *diverges then* $\sum_{n\geqslant 0} a_n$ *diverges.*

Proof —

1) *Let's assume there is* $l \in I\!R$ *such that* $\lim_{n\to+\infty}\frac{a_n}{b_n} = l$*. By the definition of the limit, we have:*

$$\forall \epsilon > 0, \exists N \in I\!N, \forall n \in I\!N : n \geqslant N \Rightarrow \left|\frac{a_n}{b_n} - l\right| < \epsilon$$

So, for all $n \geqslant N$*, we have:*

$$l - \epsilon < \frac{a_n}{b_n} < l + \epsilon,$$

Thus,

$$a \leqslant \frac{a_n}{b_n} \leqslant b,$$

where $a = l - \epsilon$ *and* $b = l + \epsilon$*. By choosing* $0 < \epsilon < l$*, we obtain* $a > 0$ *and* $b > 0$*. Therefore, applying the second version of the comparison theorem, we deduce that the two series have the same nature.*

2) A reasoning similar to the one above gives the result. Indeed, in this case, we have

$$\forall \epsilon > 0, \exists N \in I\!N, \forall n \in I\!N : n \geqslant N \Rightarrow \left| \frac{a_n}{b_n} \right| < \epsilon.$$

Thus,

$$a_n < \epsilon b_n$$

Now, by hypothesis, $\sum\limits_{n \geqslant 0} b_n$*. converges. Therefore, according to the first comparison criterion,* $\sum\limits_{n \geqslant 0} a_n$ *also converges.*

3) The result follows from the definition of the limit in the case where $l = +\infty$ *and from the application of the second comparison theorem.*

Example 1.2.6. *We want to study the convergence of the series* $\sum\limits_{n \geqslant 0} \sin\left(\frac{1}{n}\right)$*.*

We already know that $\sum\limits_{n \geqslant 0} \frac{1}{n}$ *is a divergent series. Let's set* $a_n = \sin\left(\frac{1}{n}\right)$ *and* $b_n = \frac{1}{n}$*. Since*

$$\lim_{n \to +\infty} \frac{a_n}{b_n} = 1 > 0,$$

the series have the same nature.

Remark 1.2.7.. —

If $l = 1$*, we say* $a_n \sim b_n$ *as* $n \to +\infty$ *and the series* $\sum\limits_{n \geqslant 0} a_n$ *and* $\sum\limits_{n \geqslant 0} b_n$ *will have the same nature.*

Theorem 1.2.8.. — ***(Cauchy's Rule)***

Let $\sum_{k\geqslant 0} a_n$ *be a series with positive real terms. Suppose there exists* $l \in I\!R$ *such that* $\lim_{n\to+\infty} \sqrt[n]{a_n} = l$.

- *If* $l < 1$, *then* $\sum_{n\geqslant 0} a_n$ *converges.*

- *If* $l > 1$, *then* $\sum_{n\geqslant 0} a_n$ *diverges.*

PROOF — *By the definition of the limit, we have:*

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = l \Leftrightarrow \forall \epsilon > 0, \exists N \in I\!N, \quad \forall n \geqslant N \quad |\sqrt[n]{a_n} - l| < \epsilon.$$

Thus, $l - \epsilon < \sqrt[n]{a_n} < l + \epsilon$.

- *If* $l < 1$, *we choose* ϵ_0 *such that* $l_0 = \epsilon_0 + l < 1$. *Thus,* $\sqrt[n]{a_n} < l + \epsilon_0 = l_0$ *implies* $a_n < {l_0}^n$, *and since* $\sum_{n\geqslant 0} {l_0}^n$ *is a convergent geometric series, according to the first comparison criterion,* $\sum_{n\geqslant 0} a_n$ *converges.*

- *If* $l > 1$, *we choose* ϵ_1 *such that* $l_1 = l - \epsilon_1 > 1$. *Thus,* $\sqrt[n]{a_n} > l - \epsilon_1 = l_1$ *implies* $a_n > {l_1}^n$, *and since* $\sum_{n\geqslant 0} {l_1}^n$ *is a divergent geometric series, according to the first comparison criterion,* $\sum_{n\geqslant 0} a_n$ *diverges.*

Example 1.2.9. *We want to study the convergence of the series* $\sum_{n\geqslant 0} \frac{1}{n^n}$.

Let's define $a_n = \frac{1}{n^n}$. *We have* $\sqrt[n]{a_n} = \frac{1}{n} \to 0 < 1$ *as* $n \to +\infty$. *Therefore, the series converges.*

Remark 1.2.10.. —

If $l = 1$, *we cannot conclude anything about the nature of the series.*

Theorem 1.2.11.. — ***(Rule of d'Alembert)***

Let $\sum_{k\geqslant 0} a_n$ *be a series with positive real terms. Suppose there exists* $l \in I\!R$ *such that* $\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = l$.

1) If $l < 1$, then $\sum_{n\geqslant 0} a_n$ converges.

2) If $l > 1$, then $\sum_{n\geqslant 0} a_n$ diverges.

PROOF — *By the definition of the limit, we have:*

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = l \Leftrightarrow \forall\epsilon > 0, \exists N \in I\!N, \quad \forall n \geqslant N \quad \left|\frac{a_{n+1}}{a_n} - l\right| < \epsilon.$$

Thus, for all $n \in I\!N$ we have

$$l - \epsilon < \frac{a_{n+1}}{a_n} < l + \epsilon.$$

1) If $l < 1$, we choose $\epsilon > 0$ such that $l_0 = l + \epsilon < 1$. Then,

$$a_{n+1} \leqslant l_0 a_n \quad \forall n \in I\!N.$$

Thus, for all $n \geqslant N$, we deduce

$$a_n \leqslant {l_0}^{n-N} a_N.$$

Therefore, for all $n \geqslant N$

$$a_n \leqslant {l_0}^n {l_0}^{-N} = {l_0}^n c,$$

where c is a strictly positive constant.

Since $0 < l_0 < 1$, $\sum_{n\geqslant 0} {l_0}^n$ converges, and according to the first comparison theorem, $\sum_{n\geqslant 0} a_n$ also converges.

2) If $l > 1$, we choose $\epsilon > 0$ such that $l_0 = l - \epsilon > 1$. Then, for all $n \geqslant N$, we have

$$a_n \geqslant {l_0}^{n-N} a_N.$$

Since $l_0 > 1$, we have

$$a_{n+1} \geqslant a_n \quad \forall n \geqslant N.$$

Thus, (a_n) is increasing for all $n \geqslant N$ with $a_N > 0$.
Therefore,

$$\lim_{n\to+\infty} a_n \neq 0 \quad \textit{and} \quad \sum_{n\geqslant 0} a_n \textit{ diverges.}$$

Example 1.2.12. *We wish to study the nature of the series* $\sum_{n\geqslant 0}\frac{1}{(n+1)!}$, *which is a series with positive terms.*
Let $a_n = \frac{1}{(n+1)!}$.
We have $\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \to 0 < 1$. *Thus* $\sum_{n\geqslant 0} a_n$ *converges.*

Remark 1.2.13.. —

1) *If* $l = 1$ *then nothing can be deduced.*

2) *If* $\lim_{n\to+\infty}\frac{a_{n+1}}{a_n} = l$, *then* $\lim_{n\to+\infty}\sqrt[n]{a_n} = l$, *implying that if the ratio test (d'Alembert's test) leads to a limit equal to 1, the same holds true for the Cauchy's root test.*

Lemme 1.2.1. *Let* $\sum_{n\geqslant 0} a_n$ *and* $\sum_{n\geqslant 0} b_n$ *be two series with positive terms.*
Assume that $\forall n \in I\!N$, $\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$.

1) *If* $\sum_{n\geqslant 0} b_n$ *converges then* $\sum_{n\geqslant 0} a_n$ *converges.*

2) *If* $\sum_{n\geqslant 0} a_n$ *diverges then* $\sum_{n\geqslant 0} b_n$ *diverges.*

PROOF — *By hypothesis we have*

$$\frac{a_{n+1}}{b_{n+1}} \leqslant \frac{a_n}{b_n} \leqslant ... \leqslant \frac{u_0}{v_0} = c$$

With $c > 0$. *Thus,*

$$a_n \leqslant cb_n \quad \forall n \in \mathbb{N}.$$

According to the first comparison criterion, we obtain the result.

Theorem 1.2.14.. — ***(Duhamel's Rule)***

Let $\sum_{n\geqslant 0} a_n$ *be a series with positive terms. Suppose*

$$\frac{a_{n+1}}{a_n} = 1 - \frac{b}{n} + o\left(\frac{1}{n}\right)$$

- *If* $b > 1$ *then* $\sum_{n\geqslant 0} a_n$ *converges.*

- *If* $b < 1$ *then* $\sum\limits_{n \geqslant 0} a_n$ *diverges.*

PROOF —

- *If* $b > 1$, *there exists* c *such that* $b > c > 1$. *Let* $b_n = \frac{1}{n^c}$.

  We have:

  $$\frac{b_{n+1}}{b_n} = \left(\frac{n}{n+1}\right)^c = \left(1 + \frac{1}{n}\right)^{-c}$$

  By a first development limited to the vicinity of the infinite order 1 has:

  $$\left(1 + \frac{1}{n}\right)^{-c}) = 1 - \frac{b}{n} + o\left(\frac{1}{n}\right)$$

  Now $b > c$ *thus* $-\frac{b}{n} < -\frac{c}{n}$. *Therefore,*

  $$1 - \frac{b}{n} + o\left(\frac{1}{n}\right) < 1 - \frac{c}{n} + o\left(\frac{1}{n}\right).$$

  So

  $$\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}.$$

  Therefore, since the series $b_n = \frac{1}{n^c}$ *converges, according to the previous lemma we deduce that* $\sum\limits_{n \geqslant 0} a_n$ *also converges.*

- *If* $b < 1$, *let us put here* $\sum\limits_{n \geqslant 0} b_n = \sum\limits_{n \geqslant 0} \frac{1}{n}$. *So we have here:*

  $$\frac{b_{n+1}}{b_n} = \left(\frac{n}{n+1}\right) = \frac{1}{1 + \frac{1}{n}}$$

  And since $b < 1$ *then*

  $$1 - \frac{b}{n} + o\left(\frac{1}{n}\right\rangle - \frac{1}{n} + 1 + o\left(\frac{1}{n}\right).$$

  That gives

  $$\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}.$$

  And then, according to the previous lemma, since $\sum\limits_{n \geqslant 0} b_n$ *diverges then* $\sum\limits_{n \geqslant 0} a_n$ *diverges too.*

Example 1.2.15. *We wish to study the nature of the series* $\sum_{n \geqslant 0} \frac{(2n)!}{2^{2n}(n!)^2}$.

Let $a_n = \frac{(2n)!}{2^{2n}(n!)^2}$. *We have:*

$$\frac{a_{n+1}}{a_n} = \frac{2n+1}{2n+2} = \left(1 + \frac{1}{2n}\right)\left(1 - \frac{1}{n} + o\left(\frac{1}{n}\right)\right)$$

We then have $b = \frac{1}{2} < 1$, *so the series diverges.*

Remark 1.2.16.. —

Duhamel's rule applies when $\lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = 1$.

Theorem 1.2.17.. — ***(Integral Criterion)***

Let $f : [1, +\infty[\to \mathbb{R}$ *be continuous, decreasing, and positive. Define* $f(n) = a_n$ *for all* $n \geqslant 1$. *then:*

$\sum_{n \geqslant 1} a_n$ *converges if and only if* $\lim_{T \to +\infty} \int_1^T f(x)\,dx$ *converges.* PROOF — *Since* f *is decreasing, for all* $x \in [n, n+1]$

$$f(n+1) \leqslant f(x) \leqslant f(n)$$

Thus,

$$\int_n^{n+1} f(n+1)\,dx \leqslant \int_n^{n+1} f(x)\,dx \leqslant \int_n^{n+1} f(n)\,dx \quad \forall n \geqslant 1.$$

Hence,

$$f(n+1)\int_n^{n+1} dx \leqslant \int_n^{n+1} f(x)\,dx \leqslant f(n)\int_n^{n+1} dx \quad \forall n \geqslant 1.$$

$$f(n+1) \leqslant \int_n^{n+1} f(x)\,dx \leqslant f(n) \quad \forall n \geqslant 1$$

So,

$$\text{For } n = 1, \quad f(2) \leqslant \int_1^2 f(x)\,dx \leqslant f(1),$$

$$\text{For } n = 2, \quad f(3) \leqslant \int_2^3 f(x)\,dx \leqslant f(2),$$

$$\dots$$

$$\text{For } n = 1, \quad f(n) \leqslant \int_{n-1}^{n} f(x)\,dx \leqslant f(n-1)$$

By summation, we have:

$$f(2) + f(3) + \ldots + f(n) \leqslant \int_1^n f(x)\,dx \leqslant f(1) + f(2) + \ldots + f(n-1)$$

This yields

$$S_n - a_1 \leqslant \int_1^n f(x)\,dx \leqslant S_{n-1}$$

Assuming $\lim\limits_{T\to+\infty} \displaystyle\int_1^T f(x)\,dx$ *converges, then* $\int_1^T f(x)\,dx$ *is bounded, so* (S_n) *is bounded. Since it is increasing* ($f(n) \geqslant 0$)*, it converges. Therefore,* $\sum\limits_{n\geqslant 1} a_n$ *converges.*

If $lim_{T\to+\infty} \int_1^T f(x)\,dx$ *diverges, then* S_n *diverges, so* $\sum\limits_{n\geqslant 1} a_n$ *diverges.*

Example 1.2.18. *Consider the Riemann series* $\sum\limits_{n\geqslant 1} \dfrac{1}{n^p}$ *with* $p > 0$.

This series converges if and only if $p > 1$.

Here's why, define $f(x) = \frac{1}{x^p}$ *for all* $x \geqslant 1$.

It is clear that f *is positive and continuous on* $[0, +\infty[$.

We have $f'(x) = -px^{-p-1} < 0$. *Therefore,* $\sum\limits_{T\geqslant 1} \dfrac{1}{n^p}$ *and* $\lim\limits_{T\to+\infty} \displaystyle\int_1^T f(x)\,dx = \dfrac{1}{x^p}$ *have the same nature.*

We have

$$\int \frac{dx}{x^p} = \frac{1}{(1-p)x^{p-1}} \quad and \quad \int \frac{dx}{x^p} = ln(x).$$

Then

$$\lim_{T\to+\infty} \int_1^T f(x)\,dx = -\frac{1}{-p+1} \quad If \quad p > 1,$$

$$lim_{T\to+\infty} \int_1^T f(x)\,dx = +\infty \quad If \quad 0 < p \leqslant 1,$$

Thus, $\lim\limits_{T\to+\infty} \displaystyle\int_1^T f(x)\,dx$ *converges if* $p > 1$.

Thus $\sum\limits_{n\geqslant 1} \dfrac{1}{n^p}$ *converges if* $p > 1$.

1.3 Alternating Series

Definition 1.3.1. *A series of the form* $\sum\limits_{n\geqslant 1} (-1)^n a_n$, *with* $a_n \geqslant 0$ *is called an alternating series.*

Theorem 1.3.2.. — ***(Leibnitz Criterion)***

Let $\sum_{n\geqslant 0}(-1)^n a_n$ *be an alternating series.*
If the sequence (a_n) *is decreasing and tends to* 0 *as* n *approaches infinity then* $\sum_{n\geqslant 1}(-1)^n a_n$ *converges.* Proof — *This criterion is a particular case of a criterion called Abel's criterion, which will be stated and proved later.*

Example 1.3.3. *Consider the series* $\sum_{n\geqslant 1}\frac{(-1)^n}{n}$.
Let $a_n = \frac{1}{n}$.
It is evident that (a_n) *is positive, decreasing, and tends to* 0.

Thus, according to Leibnitz's criterion, the series converges.

Corollary 1.3.1. *Let* $\sum_{n\geqslant 0}(-1)^n a_n$ *be a convergent alternating series, then:*

$$\left|\sum_{n\geqslant k}(-1)^n a_n\right| \leqslant a_{k+1} \quad \forall k \in \mathbb{N}$$

Remark 1.3.4.. —

The previous corollary provides an estimation of the remainder of an alternating series.

Example 1.3.5. *Consider the series* $\sum_{n\geqslant 0}\frac{(-1)^n}{n}$. *We want to determine the integer* n *such that the sum approximates the series with an error less than* 10^{-2}. *This series being an alternating series satisfies the previous corollary, so we have:*

$$\left|\sum_{n\geqslant k}(-1)^n a_n\right| \leqslant a_{k+1}, \quad \forall k \in \mathbb{N}$$

To find the natural number n, *it suffices to satisfy the following inequality:*

$$a_{k+1} \leqslant 10^{-2}, \quad \forall k \in \mathbb{N}.$$

Thus, $n = 99$. *Therefore,* $\sum_{n\geqslant 0}\frac{(-1)^n}{n} = \sum_{n=1}^{99}\frac{(-1)^n}{n}$ *with an error of* 10^{-2}.

1.4 Series with Arbitrary Terms

Definition 1.4.1. *A series* $\sum_{n\geqslant 0} a_n$ *with arbitrary real terms is said to be absolutely convergent if the series* $\sum_{n\geqslant 0} |a_n|$ *converges.*

Remark 1.4.1.. —

The series $\sum_{n\geqslant 0} |a_n|$ *is a series with positive terms, so the study of absolute convergence can be done using criteria for series with positive terms.*

Example 1.4.2. *We want to study the absolute convergence of the series* $\sum_{n\geqslant 1} \frac{(-1)^n}{n^2}$.
Let $a_n = \frac{(-1)^n}{n^2}$.
We have $\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$*, which is a convergent Riemann series. Therefore, the series* $\sum_{n\geqslant 1} \frac{(-1)^n}{n^2}$ *absolutely converges.*

Theorem 1.4.3.. — *A*

series that is absolutely convergent is convergent.

$$\sum_{n\geqslant 0} |a_n| \quad \text{converges} \Rightarrow \sum_{n\geqslant 0} a_n \quad \text{converges.}$$

Proof — *Assume* $\sum_{n\geqslant 0} |a_n|$ *converges. Then the partial sum* (S_n) *associated with* $\sum_{n\geqslant 0} |a_n|$ *converges, thus it is Cauchy. Hence,*

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geqslant N, \sum_{k=n+1}^{m} |a_n| < \epsilon.$$

And since for all n, m *we have*

$$\left|\sum_{k=n+1}^{m} a_n\right| \leqslant \sum_{k=n+1}^{m} |a_n|,$$

we conclude that

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geqslant N \Rightarrow \left|\sum_{k=n+1}^{m} a_n\right| < \epsilon.$$

Therefore, the partial sum (S'_n) *associated with* $\sum_{n\geqslant 0} a_n$ *is Cauchy, thus it converges.*

Example 1.4.4. *We want to study the convergence of the series* $\sum_{n\geqslant 1}\frac{(-1)^n}{n^2}$*. The absolute convergence of this series was shown in the previous example, so we can deduce that this series converges.*

1.4.1 Semi-convergence

Definition 1.4.2. *A semi-convergent series is a convergent series that does not converge absolutely.*

Example 1.4.5. *Consider the series* $\sum_{n\geqslant 1}\frac{(-1)^n}{n}$*. The convergence of this series has already been shown by the Leibniz criterion. Now, we study absolute convergence.*

$$\sum_{n\geqslant 1}\left|\frac{(-1)^n}{n}\right| = \sum_{n\geqslant 1}\frac{1}{n}.$$

This series is the divergent harmonic series, so the series converges but does not converge absolutely.

Theorem 1.4.6.. — ***(Abel's Criterion)***

Let $\sum_{n\geqslant 0} a_nb_n$ *be a numerical series such that:*

1) The sequence $(a_n)_{n\in\mathbb{N}}$ *is positive, strictly increasing, and* $a_n \to 0$ *as* $n \to +\infty$*.*

2) There exists $M > 0$ *such that for all* $n \in \mathbb{N}$, $\left|\sum_{k\geqslant 0} b_k\right| < M$*.*

Then $\sum_{n\geqslant 0} a_nb_n$ *converges.*

Proof — *The idea of the proof is to perform a change in summation. For each* $n > 0$*, let* $B_n = b_0 + \ldots + b_n$*. By hypothesis, the sequence* (B_n) *is bounded. We write the partial sums of the series* $\sum_{n\in\mathbb{N}} a_nb_n$ *as follows:*

$$S_n = a_0b_0 + a_1b_1 + \ldots + a_{n-1}b_{n-1} + a_nb_n$$

$$= a_0B_0 + a_1(B_1 - B_0) + \ldots + a_{n-1}(B_{n-1} - B_{n-2}) + a_n(B_n - B_{n-1})$$

$$= B_0(a_0 - a_1) + B_1(a_1 - a_2) + \ldots + B_{n-1}(a_{n-1} - a_n) + B_na_n.$$

Since (B_n) *is bounded and* $a_n \to 0$, *the last term* $B_n a_n$ *tends to* 0.

Now, we show that the series $\sum_{k\in\mathbb{N}} B_{k-1}(a_{k-1}-a_k)$ *is absolutely convergent. Indeed:*

$$|B_k(a_k-a_{k+1})| = |B_k|\,|(a_k-a_{k+1})| \leqslant M\,|(a_k-a_{k+1})|\,,$$

since the sequence (a_k) *is a sequence of positive real numbers, decreasing, and* $|B_k|$ *is bounded by* M. *Moreover,*

$$M(a_0-a_1)+\ldots+M(a_n-a_{n+1}) = M(a_0-a_{n+1}),$$

which tends to Ma_0 *as* (a_k) *tends to* 0. *The series* $\sum M(a_k-a_{k+1})$ *therefore converges to a series of comparison. So the series*

$$\sum |B_k(a_k-a_{k+1})|$$

is convergent, therefore, $\sum B_k(a_k-a_{k+1})$ *is convergent, therefore, the suite* (S_n) *is convergent, which proves that the series* $\sum a_k b_k$ *is convergent.*

Example 1.4.7. *Consider* $\sum_{n\geqslant 1}\frac{\sin(n)}{n}$. *Let* $a_n=\frac{1}{n}$ *and* $b_n=\sin(n)$. *It is clear that* a_n *is positive, decreasing, and tends to* 0.

$$\left|\sum_{k=1}^{n}\sin(k)\right| = \left|\frac{\sin(\frac{n}{2})\sin(\frac{n+1}{2})}{\sin(\frac{1}{2})}\right| \leqslant \frac{1}{\sin(\frac{1}{2})}.$$

According to Abel's criterion, the series $\sum_{n\geqslant 1} a_n b_n$.

1.5 Exercises:

Exercise 01:

Study the nature of the following series using partial sums:

$$\sum_{n\geqslant 1}\frac{1}{(2n-1)(2n+1)},\quad \sum_{n\geqslant 1}\frac{3-n}{n(n+1)(n+2)},\quad \sum_{n\geqslant 3}\frac{2n-1}{n(n^2-4)}.$$

Exercise 02:

Determine the nature of the following series using convergence criteria:

$$1)\sum_{n\geqslant 1}\frac{1}{n\cos^2(n)},\quad 2)\sum_{n\geqslant 2}\frac{1}{(\ln(n))^n},\quad 3)\sum_{n\geqslant 1}\frac{2^n}{3^{n-2}},4)\sum_{n\geqslant 1}\frac{\tan^n\left(\frac{\pi}{7}\right)}{3^{n-2}},\quad 5)\sum_{n\geqslant 2}(-1)^n\frac{n+1}{n^2-1},$$

$$6)\sum_{n\geqslant 1}n\sin\left(\frac{1}{n}\right),\quad 7)\sum_{n\geqslant 1}\frac{2^n+3^n}{n^2+\ln(n)+5^n},8)\sum_{n\geqslant 1}\Big(1-\cos\left(\frac{\pi}{n}\right))(\ln(n)\Big)^{2024},$$

$$9)\sum_{n\geqslant 1}\frac{1}{n^{1+1/\sqrt{n}}},10)\sum_{n\geqslant 0}a^n n!\quad \textit{where } a\in\mathbb{R},\quad 11)\sum_{n\geqslant 12}\frac{2^n}{(2^n+n)^n},\quad 12)\sum_{n\geqslant 1}\frac{1}{(n+1)\ln(n+1)}$$

Exercise 03:

Study the convergence of the following series, specifying which are convergent and which are semi-convergent:

$$1)\sum_{n\geqslant 1}\frac{(-1)^{n-1}}{2n-1},\quad 2)\sum_{n\geqslant 1}(-1)^n\left(\frac{2n+1}{3n+1}\right)^n,\quad 3)\sum_{n\geqslant 1}(-1)^n\sin\left(\frac{1}{n}\right),$$

$$4)\sum_{n\geqslant 1}(-1)^n\frac{\sqrt{n}}{n},\quad 5)\sum_{n\geqslant 1}(-1)^n\frac{1}{\ln(\sqrt{n}+1)},\quad 6)\sum_{n\geqslant 0}\sin(n),$$

$$7)\sum_{n\geqslant 1}\frac{\cos(\sqrt{n})}{n\sqrt{n}}.$$

Exercise 04:

Consider the function f defined by

$$f(x)=\frac{\ln(x)}{x^\alpha},\quad x>0,\alpha>0.$$

1) *Establish the variation table of the function f.*
2) *Study the nature of the numerical series $\sum_{n\geqslant 1}\frac{(-1)^n}{n^\alpha\ln(n)}$ with $\alpha>0$.*
3) *Deduce the nature of the series $\sum_{n\geqslant 1}\ln(n)\ln\left(1+\frac{(-1)^n}{n^\alpha}\right)$ with $\alpha\geqslant\frac{1}{2}$.*

Exercise 05:

Show that the series $\sum_{n\geqslant 1} \frac{(-1)^n}{\sqrt{n}}$ *and* $\sum_{n\geqslant 1} \left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right)$ *are not of the same nature, despite having equivalent general terms.*

Exercise 06:

Consider the sequence u_n *defined by*

$$u_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}.$$

1) *Show that* $u_n \sim \ln(n)$ *as* $n + \infty$.

2) *Let* $v_n = u_n - \ln(n)$ *and* $w_n = v_{n+1} - v_n$. *Study the nature of the series* $\sum\limits_{n\geqslant 1} w_n$ *and deduce the nature of the sequence* $(v_n)_n$.

Exercise 07:

Let u_n *be the numerical sequence defined by*

$$u_n = \frac{n^n e^{-n}\sqrt{n}}{n!}.$$

1) *Using a Taylor expansion, study the convergence of the series with general term* $v_n = \ln\left(\frac{u_{n+1}}{u_n}\right)$.

2) *Deduce the existence of a constant* $\alpha > 0$ *such that in the vicinity of infinity,*

$$n! \sim n^n e^{-n}\sqrt{n} \cdot \alpha.$$

Exercise 08:

Let $\sum\limits_{n\geqslant 1} u_n$ *and* $\sum\limits_{n\geqslant 1} v_n$ *be two series such that* $\sum\limits_{n\geqslant 1} u_n^2$ *and* $\sum\limits_{n\geqslant 1} v_n^2$ *converge.*

1) *Show that* $\sum\limits_{n\geqslant 1} u_n v_n$ *also converges.*

2) *Deduce that if the sequence* $(u_n)_n$ *is positive and the series* $\sum\limits_{n\geqslant 1} \frac{1}{1+n^2 u_n}$ *converges, then the series* $\sum\limits_{n\geqslant 1} u_n$ *diverges.*

Exercise 09

We want to study, depending on the values of $\alpha, \beta \in \mathbb{R}$*, the convergence of the series with general term*

$$u_n = \frac{1}{n^\alpha (\ln n)^\beta}.$$

1. *Show that the series converges if* $\alpha > 1$ *.*

2. *Analyze the convergence of the series if* $\alpha < 1$*.*

3. ***Case*** $\alpha = 1$***:***
   Let $T_n = \int_2^n \frac{dx}{x(\ln x)^\beta}$*.*

   (a) *If* $\beta \leqslant 0$*, demonstrate that the series* $\sum_{n \geqslant 2} u_n$ *diverges.*

   (b) *If* $\beta > 1$*, show that the sequence* (T_n) *is bounded.*

   (c) *If* $0 < \beta \leqslant 1$*, demonstrate that the sequence* (T_n) *tends to* $+\infty$*.*

   (d) *Conclude on the nature of the series with general term* u_n *when* $\alpha = 1$*.*

1.6 Solution:

Solution to Exercise 1

Study of the nature of the following series using partial sums:

1) First, observe that

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)},$$

so by grouping,

$$\begin{aligned}\sum_{n=1}^{k} \frac{1}{(2n-1)(2n+1)} &= (\frac{1}{2} - \frac{1}{6}) + (\frac{1}{6} - \frac{1}{10}) + \ldots + (\frac{1}{2(2k-3)} - \frac{1}{2(2k-1)}) \\ &\quad + (\frac{1}{2(2k-1)} - \frac{1}{2(2k+1)}) \\ &= \frac{1}{2} - \frac{1}{2(2k+1)} \to \frac{1}{2}\end{aligned}$$

as $k \to +\infty$. *Therefore, the sum of the series converges.*

2) First, observe that

$$\frac{3-n}{n(n+1)(n+2)} = \frac{1}{2}(\frac{3}{n} - \frac{8}{n+1} + \frac{5}{n+2}),$$

so by grouping,

$$\begin{aligned}\sum_{n=1}^{k} \frac{3-n}{n(n+1)(n+2)} &= \frac{1}{2}\Big((3 - \frac{8}{2} + \frac{5}{3}) + (\frac{3}{2} - \frac{8}{3} + \frac{5}{4}) \\ &\quad + (\frac{3}{3} - \frac{8}{4} + \frac{5}{5}) + \ldots + (\frac{3}{k-2} - \frac{8}{k-1} + \frac{5}{k}) \\ &\quad + (\frac{3}{k-1} - \frac{8}{k} + \frac{5}{k+1}) + (\frac{3}{k} - \frac{8}{k+1} + \frac{5}{k+2})\Big) \\ &= \frac{1}{2}\Big(3 - 4 + \frac{3}{2} - \frac{3}{k+1} + \frac{5}{k+2}\Big) \to \frac{1}{4}\end{aligned}$$

as $k \to +\infty$. *Therefore, the sum of the series converges.*

3)

$$\begin{aligned}
\sum_{n=3}^{k}\frac{2n-1}{n(n^2-4)} &= \sum_{n=3}^{k}\left(\frac{1}{4n}+\frac{3}{8(n-2)}-\frac{5}{8(n+2)}\right)\\
&= \sum_{n=3}^{k}\frac{1}{4n}+\sum_{n=1}^{k-2}\frac{3}{8n}-\sum_{n=5}^{k+2}\frac{5}{8n}\\
&= \frac{1}{4}\left(\frac{1}{3}+\frac{1}{4}+\frac{3}{2}+\frac{3}{4}+\frac{1}{2}+\frac{3}{8}\right)-\frac{3}{8(k-1)}-\frac{3}{8k}-\frac{5}{8(k+1)}-\frac{5}{8(k+2)}\\
&\to \frac{89}{96}
\end{aligned}$$

as $k \to +\infty$. Therefore, the sum of the series converges.

Solution to Exercise 2

Study the nature of the following series using convergence criteria:
1)

$$\frac{1}{n\cos^2(n)} \geqslant \frac{1}{n}.$$

This is a series with positive terms greater than $\frac{1}{n}$, which is the general term of a divergent Riemann series with $\alpha = 1 \leqslant 1$. Thus, the series $\sum_{n\geqslant 1}\frac{1}{n\cos^2(n)}$ diverges.

2) $\sum_{n\geqslant 1}\frac{1}{(\ln(n))^n}$ *is a series with positive terms.*

$$\sqrt[n]{\frac{1}{(\ln(n))^n}} = \frac{1}{(\ln(n))} \underset{n\longrightarrow+\infty}{\longrightarrow} 0.$$

By Cauchy's root test, $0 < 1$, hence the series converges.

3) $\frac{2^n}{3^{n-2}} = 4 \times \left(\frac{2}{3}\right)^n$*, is the general term of a geometric series with ratio in* $]-1,1[$*, hence the series converges.*

4) $\frac{\tan^n(\frac{\pi}{7})}{3^{n-2}} \leqslant \frac{1}{3^{n-2}} = 9 \times \left(\frac{1}{3}\right)^n$*, is the general term of a geometric series with ratio* *in* $]-1,1[$, *which converges. By comparison test, the series* $\sum_{n\geqslant 1}\frac{\tan^n(\frac{\pi}{7})}{3^{n-2}}$ *converges.*

5)

$$\sum_{n\geqslant 2}(-1)^n\frac{n+1}{n^2-1}=\sum_{n\geqslant 2}(-1)^n\frac{1}{n-1}=\sum_{n\geqslant 2}(-1)^n a_n,$$

where $a_n>0$, $a_n\to 0$ *as* $n\to+\infty$, *and the sequence* (a_n) *is decreasing. By the Leibniz criterion, the alternating series converges.*

6) We have $\lim_{n\longrightarrow+\infty} n\sin(\frac{1}{n})=1$ *(recall* $\sin x \underset{0}{\sim} x$*), and thus the series* $\sum_{n\geqslant 1} n\sin(\frac{1}{n})$ *diverges (its general term does not tend to* 0*).*

7) $\sum_{n\geqslant 1}\frac{2^n+3^n}{n^2+\ln(n)+5^n}$ *is a series with positive terms.*

$$\frac{2^n+3^n}{n^2+\ln(n)+5^n}\underset{+\infty}{\sim}\frac{3^n}{5^n}=\left(\frac{3}{5}\right)^n.$$

$\left(\frac{3}{5}\right)^n$ *is the general term of a convergent geometric series* $(-1<k<1)$, *hence the series converges.*

8) $\sum_{n\geqslant 1}(1-\cos(\frac{\pi}{n}))(\ln(n))^{2024}$ *is a series with positive terms (*$0<1-\cos(\frac{\pi}{n})\leqslant 2$*). Using the cosine approximation,* $1-\cos x\underset{0}{\sim}\frac{x^2}{2}$, *we see that*

$$(1-\cos(\frac{\pi}{n}))(\ln(n))^{2024}\sim\frac{\pi^2}{2n^2}(\ln(n))^{2024},$$

and

$$\lim_{n\to+\infty}\frac{\frac{(\ln(n))^{2024}}{n^2}}{\frac{1}{n^3}}=\lim_{n\to+\infty}\frac{(\ln(n))^{2024}}{n}=0,$$

since $\sum_{n\geqslant 1}\frac{1}{n^3}$ *converges, then* $\sum_{n\geqslant 1}\frac{(\ln(n))^{2024}}{n^2}$ *converges.*

Thus, $\sum_{n\geqslant 1}(1-\cos(\frac{\pi}{n}))(\ln(n))^{2024}$ *converges.*

9) $\sum_{n\geqslant 1}\frac{1}{n^{1+\frac{1}{\sqrt{n}}}}$ *is a series with positive terms.*

$$a_n=\frac{1}{n^{1+\frac{1}{\sqrt{n}}}}=\frac{1}{n}n^{-\frac{1}{\sqrt{n}}}=\frac{1}{n}\exp^{-\frac{\ln(n)}{\sqrt{n}}}.$$

Since $\lim\limits_{n\longrightarrow+\infty}\dfrac{\ln(n)}{\sqrt{n}}=0$*, we have* $\lim\limits_{n\longrightarrow+\infty}\exp^{-\frac{\ln(n)}{\sqrt{n}}}=1$*. This shows* $a_n\underset{+\infty}{\sim}\dfrac{1}{n}$*, which is the general term of a divergent Riemann series with* $\alpha=1\leqslant 1$*. Therefore,* $\sum\limits_{n\geqslant 1}\dfrac{1}{n^{1+\frac{1}{\sqrt{n}}}}$ *diverges.*

10) $\sum\limits_{n\geqslant 0}a^n n!$, $a\in\mathbb{R}$*. For* $a>0$,

$$\lim_{n\to+\infty}\frac{a^{n+1}(n+1)!}{a^n n!}=\lim_{n\to+\infty}a(n+1)=+\infty>1.$$

Thus, by the ratio test (d'Alembert's criterion), $\sum\limits_{n\geqslant 0}a^n n!$ *diverges.*

For $a=0$, $\sum\limits_{n\geqslant 0}a^n n!$ *converges.*

For $a<0$*, let* $a=-b$ *with* $b>0$*, the series becomes* $\sum\limits_{n\geqslant 0}(-1)^n b^n n!$.

Since $\sum\limits_{n\geqslant 0}\dfrac{1}{(-1)^n b^n n!}$ *is absolutely convergent by d'Alembert's criterion, it converges. This implies* $\lim\limits_{n\to+\infty}\dfrac{1}{(-1)^n b^n n!}=0$*, hence* $\lim\limits_{n\to+\infty}(-1)^n b^n n!\neq 0$*. Therefore,* $\sum\limits_{n\geqslant 0}(-1)^n b^n n!$ *diverges.*

11) $\sum\limits_{n\geqslant 2}\dfrac{2^n}{(2^n+n)^n}$ *is a series with positive terms.*

$$\sqrt[n]{\frac{2^n}{(2^n+n)^n}}=\frac{2}{2^n+n}\underset{n\longrightarrow+\infty}{\longrightarrow}0.$$

By Cauchy's root test, $0<1$*, hence the series converges.*

12) We can apply the integral test; let $f(x)=\dfrac{1}{(x+1)\ln(x+1)}$*, a positive and continuous function for all* $x\geqslant 1$*. This function is clearly decreasing (*$f'(x)=\dfrac{-1-\ln(1+x)}{(\ln(x+1))^2}$*) and tends to* 0 *as* $x\to\infty$*, and it is integrable because*

$$\int_1^T\frac{1}{(x+1)\ln(x+1)}dx=\left[\ln\big(\ln(1+x)\big)\right]_1^T=\ln\big(\ln(1+T)\big)-\ln\big(\ln(2)\big)\to+\infty,$$

as $T\to+\infty$*, showing that the integral diverges. Hence, the series with general term* $\dfrac{1}{(n+1)\ln(n+1)}$ *diverges.*

Solution to Exercise 3

Study the convergence of the following series, specifying those that are convergent versus semi-convergent:

1)

$$\left|\frac{(-1)^{n-1}}{2n-1}\right| = \frac{1}{2n-1} \sim \frac{1}{2n}$$

and the series $\sum_{n\geqslant 1} \frac{1}{2n}$ *diverges (Riemann series with* $\alpha = 1$*). Therefore,* $\sum_{n\geqslant 1} \left|\frac{(-1)^{n-1}}{2n-1}\right|$ *diverges, showing that the series* $\sum_{n\geqslant 1} \frac{(-1)^{n-1}}{2n-1}$ *does not converge absolutely.*

$\sum_{n\geqslant 1}(-1)^{n-1}a_n$ *converges by Leibniz's criterion since* $a_n > 0$ *and* $\lim_{n\longrightarrow+\infty} a_n = 0$, *and the sequence* (a_n) *is decreasing. Hence,* $\sum_{n\geqslant 1} \frac{(-1)^{n-1}}{2n-1}$ *is a semi-convergent series.*

2)

$$\sqrt[n]{\left|(-1)^n\left(\frac{2n+1}{3n+1}\right)^n\right|} = \sqrt[n]{\left(\frac{2n+1}{3n+1}\right)^n} = \frac{2n+1}{3n+1} \underset{n\longrightarrow+\infty}{\longrightarrow} \frac{2}{3}.$$

By Cauchy's root test, $\frac{2}{3} < 1$, *the series* $\sum_{n\geqslant 1}(-1)^n\left(\frac{2n+1}{3n+1}\right)^n$ *converges absolutely, hence converges.*

3) $\sum_{n\geqslant 1}(-1)^n \sin(\frac{1}{n})$ *is semi-convergent, since* $\left|(-1)^n\sin(\frac{1}{n})\right| = \sin(\frac{1}{n}) \underset{+\infty}{\sim} \frac{1}{n}$ *and* $\sum_{n\geqslant 1}\frac{1}{n}$ *diverges, thus* $\sum_{n\geqslant 1}\left|(-1)^n\sin(\frac{1}{n})\right|$ *diverges. However, by Leibniz's criterion,* $\sum_{n\geqslant 1}(-1)^n\sin(\frac{1}{n})$ *converges, since* $\sin(\frac{1}{n}) > 0$ *and* $\lim_{n\longrightarrow+\infty}\sin(\frac{1}{n}) = 0$, *and the sequence* $(\sin(\frac{1}{n}))$ *is decreasing.*

$\sum_{n\geqslant 1}(-1)^n\sin(\frac{1}{n})$ *is thus a semi-convergent series.*

4) $\sum_{n\geqslant 1}(-1)^n\frac{\sqrt{n}}{n} = \sum_{n\geqslant 1}\frac{(-1)^n}{\sqrt{n}}$, *does not converge absolutely (*$\sum_{n\geqslant 1}\frac{1}{\sqrt{n}}$ *diverges), but it is semi-convergent (easily verified by Leibniz's criterion).*

5) Let $f(x) = \dfrac{1}{\ln(\sqrt{x}+1)} > 0$,

$$f'(x) = -\frac{(\ln(\sqrt{x}+1))'}{(\ln(\sqrt{x}+1))^2} = -\frac{\dfrac{1}{2\sqrt{x}} \times \dfrac{1}{(\sqrt{x}+1)}}{(\ln(\sqrt{x}+1))^2} < 0.$$

Therefore, the sequence with general term $a_n = f(n)$ *is decreasing, tends to* 0, *so by Leibniz's criterion the series* $\sum\limits_{n\geqslant 1}(-1)^n \dfrac{1}{\ln(\sqrt{n}+1)}$ *converges. The series is semi-convergent because* $\left|(-1)^n \dfrac{1}{\ln(\sqrt{n}+1)}\right| = \dfrac{1}{\ln(\sqrt{n}+1)} \underset{+\infty}{\sim} \dfrac{1}{\ln(\sqrt{n})} = \dfrac{2}{\ln(n)}$ *and* $\sum\limits_{n\geqslant 1} \dfrac{2}{n^0 \ln(n)}$ *diverges (Bertrand series with* $\alpha < 1$*), thus* $\sum\limits_{n\geqslant 1}(-1)^n \dfrac{1}{\ln(\sqrt{n}+1)}$ *does not converge absolutely.*

6) $\sum\limits_{n\geqslant 0} \sin(n)$ *diverges because* $\lim\limits_{n\longrightarrow+\infty} \sin(n) \neq 0$. *Hence, this series does not converge absolutely.*

7) $\left|\dfrac{\cos(\sqrt{n})}{n\sqrt{n}}\right| \leqslant \dfrac{1}{n^{\frac{3}{2}}}$ *and* $\sum\limits_{n\geqslant 1} \dfrac{1}{n^{\frac{3}{2}}}$ *converges, thus* $\sum\limits_{n\geqslant 1} \dfrac{\cos(\sqrt{n})}{n\sqrt{n}}$ *converges absolutely. Therefore, the series converges.*

Solution to Exercise 4

Let the function f *be defined as*

$$f(x) = \frac{\ln(x)}{x^\alpha}, \quad x > 0, \quad \alpha > 0.$$

1) $f'(x) = \dfrac{1 - \alpha \ln(x)}{x^{\alpha+1}}$. *The variation table of the function* f.

x	0		$e^{\frac{1}{\alpha}}$		$+\infty$
$f'(x)$		$+$		$-$	
$f(x)$	$-\infty$	$\nearrow$	$\frac{1}{\alpha e}$	$\searrow$	0

2) Let $a_n = f(x) = \dfrac{\ln(x)}{x^\alpha} > 0$, *for* $x \geqslant 1$. *Since* $\lim\limits_{x\longrightarrow+\infty} \dfrac{\ln(x)}{x^\alpha} = 0$ *and the function*

f is decreasing for $x > e^{\frac{1}{\alpha}}$, i.e., the sequence (a_n) is decreasing from a certain rank (for $n \geqslant [e^{\frac{1}{\alpha}}] + 1$), therefore $\sum\limits_{n \geqslant 1} \frac{(-1)^n}{n^\alpha} \ln(n)$ converges by Leibniz's criterion, for $\alpha > 0$.

3) Using the second-order Taylor expansion at zero of $t \mapsto \ln(1+t)$, we verify that

$$\ln(n)\ln\left(1 + \frac{(-1)^n}{n^\alpha}\right) = \ln(n)\left(\frac{(-1)^n}{n^\alpha} - \frac{1}{2n^{2\alpha}} + o\left(\frac{1}{n^{2\alpha}}\right)\right).$$

Furthermore, based on what we know about Bertrand series,

$$\ln(n)\left(-\frac{1}{2n^{2\alpha}} + o\left(\frac{1}{n^{2\alpha}}\right)\right)$$

is the general term of a convergent series for $\alpha > \frac{1}{2}$. Thus, the series $\sum\limits_{n \geqslant 2} \ln(n)\ln\left(1 + \frac{(-1)^n}{n^\alpha}\right)$ converges for $\alpha > \frac{1}{2}$. For $\alpha = \frac{1}{2}$, the series diverges.

Solution to Exercise 5

The series $\sum\limits_{n \geqslant 1} \frac{(-1)^n}{\sqrt{n}}$ is an alternating series that converges by the Leibniz criterion ($\frac{1}{n}$ is positive, decreasing, and tends to 0).

Let's define $U_n = \frac{(-1)^n}{\sqrt{n}}$ and $V_n = \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}$. $\frac{U_n}{V_n} = 1 + \frac{(-1)^n\sqrt{n}}{n} \to 1$ as $n \to +\infty$, $\sum\limits_{n \geqslant 1} v_n$ consists of a convergent series and the harmonic series, which diverges. Therefore, $\sum\limits_{n \geqslant 1} v_n$ diverges. Hence, the series are not of the same nature, yet $u_n \sim v_n$ as $n \to +\infty$, showing that the comparison criterion applies only to series with positive terms.

Solution to Exercise 6

1) We show that $\lim\limits_{n \to +\infty} \frac{u_n}{\ln(n)} = 1$.

According to the proof of the integral criterion:

$$\int_k^{k+1} f(t)dt \leqslant u_k \leqslant \int_{k-1}^{k} f(t)dt,$$

where $f(t) = \frac{1}{t}$ and $u_k = f(k)$.

Thus, for all $k \geqslant 1$:

$$\int_k^{k+1} \frac{dt}{t} \leqslant \frac{1}{k} \leqslant \int_{k-1}^{k} \frac{dt}{t}.$$

Summing these inequalities gives:

$$\ln(n+1) - \ln 2 \leqslant \sum_{k=2}^{n} \frac{1}{k} = u_n - 1.$$

Thus,

$$\ln(n+1) - \ln 2 + 1 \leqslant u_n \leqslant \ln n + 1.$$

Consequently,

$$\frac{\ln(n+1)}{\ln n} + \frac{-\ln 2 + 1}{\ln n} \leqslant \frac{u_n}{\ln n} \leqslant 1 + \frac{1}{\ln n}.$$

Taking the limit as $n \to +\infty$, *we obtain* $\lim\limits_{n\to+\infty} \dfrac{u_n}{\ln(n)} = 1$. *Thus,* $u_n \sim \ln n$.

2) We study the nature of $\sum w_n$ *with* $w_n = v_{n+1} - v_n$ *and* $v_n = u_n - \ln n$.
We have:

$$w_n = v_{n+1} - v_n = u_{n+1} - u_n + \ln\left(\frac{n}{n+1}\right).$$

Now,

$$u_{n+1} - u_n = \frac{1}{n+1}.$$

Therefore,

$$w_n = \frac{1}{n+1} + \ln\left(\frac{n}{n+1}\right) = \frac{1}{n+1} + \ln\left(1 - \frac{1}{n+1}\right).$$

Using a Taylor expansion of $\ln\left(1 - \dfrac{1}{n+1}\right)$ *at infinity, we get:*

$$w_n = -\frac{1}{2(n+1)^2} + o\left(\frac{1}{(n+1)^2}\right).$$

This shows that $w_n \sim -\dfrac{1}{2(n+1)^2}$ *as* ∞. *Since* $\dfrac{1}{2(n+1)^2} \sim \dfrac{1}{2n^2}$ *and* $\sum\limits_{n\geqslant 1} \dfrac{1}{2n^2}$ *converges,* $\sum\limits_{n\geqslant 1} w_n$ *converges.*

Let l *such that* $\sum\limits_{n\geqslant 1} w_n = l$. *Then,*

$$\sum_{k=1}^{n} w_k = v_{n+1} - v_1.$$

Thus,

$$\lim_{n\to+\infty}(v_{n+1}-v_n)=l.$$

This shows that $\lim\limits_{n\to+\infty} v_n = v_1 + l$. *Therefore,* $(v_n)_n$ *converges.*

Solution to Exercise 7

1) We have

$$v_n=\ln\left(\frac{u_{n+1}}{u_n}\right)=\ln\left[\left(\frac{n+1}{n}\right)^{n+\frac{1}{2}}e^{-1}\right]=(n+\frac{1}{2})\ln\left(1+\frac{1}{n}\right)-1.$$

By performing a third-order Taylor expansion of $\ln\left(1+\frac{1}{n}\right)$, *we get*

$$v_n=(n+\frac{1}{2})\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+o\left(\frac{1}{n^3}\right)\right]-1.$$

Therefore,

$$v_n=\frac{1}{12n^2}+\frac{1}{6n^3}+o\left(\frac{1}{n^3}\right).$$

Consequently,

$$v_n\sim\frac{1}{12n^2}.$$

Since $\sum\limits_{n\geqslant 1}\frac{1}{12n^2}$ *converges,* $\sum\limits_{n\geqslant 1}v_n$ *converges too.*

2) We have

$$v_n=\ln u_{n+1}-\ln u_n.$$

On the other hand, the sequence of partial sums of the series $\sum\limits_{\geqslant 1}v_n$ *is*

$$S_n=\sum_{k=1}^{n}v_k=\sum_{k=1}^{n}(\ln u_{k+1}-\ln u_k).$$

Upon calculation, we have

$$S_n=\ln u_{n+1}-\ln u_1=lnu_{n+1}+1.$$

Since the series $\sum\limits_{n\geqslant 1}v_n$ *converges, then*

$$\lim_{n\to+\infty}\ln u_{n+1}=l-1.$$

Thus,

$$\lim_{n\to+\infty} u_{n+1} = e^{l-1}.$$

Let $c = e^{l-1} > 0$, *then*

$$\lim_{n\to+\infty} u_n = c.$$

Therefore,

$$\lim_{n\to+\infty} \frac{n!}{n^n e^{-n}\sqrt{n}} = \frac{1}{c}.$$

Thus,

$$n! \sim \alpha n^n e^{-n}\sqrt{n}$$

with $\alpha = \frac{1}{c}$.

Solution to Exercise 8

We assume that $\sum_{n\geqslant 1} u_n^2$ *and* $\sum_{n\geqslant 1} v_n^2$ *converge.*

1) For all $n \in \mathbb{N}$,

$$|u_n v_n| \leqslant \frac{1}{2}(u_n^2 + v_n^2).$$

Since $\sum_{n\geqslant 1} u_n^2$ *and* $\sum_{n\geqslant 1} v_n^2$ *converge, then* $\sum_{n\geqslant 1} \frac{1}{2}(u_n^2 + v_n^2)$ *converges too. By the comparison theorem, the series* $\sum_{n\geqslant 1} |u_n v_n|$ *converges, which means* $\sum_{n\geqslant 1} u_n v_n$ *converges absolutely, hence* $\sum_{n\geqslant 1} u_n v_n$ *converges.*

2) We assume that $\sum_{n\geqslant 1} \frac{1}{1+n^2 u_n}$ *converges and that* $\sum_{n\geqslant 1} u_n$ *converges. Then for all* $n \in \mathbb{N}^*$,

$$\frac{1}{1+n^2 u_n} \sim \frac{1}{n^2 u_n} \quad n- > +\infty.$$

Let $v_n = \frac{1}{n^2 u_n}$, *according to the previous relation* $\sum_{n\geqslant 1} v_n$ *converges. Moreover, since for all* $n \in I\!N$ *we have*

$$2\sqrt{u_n v_n} \leqslant u_n + v_n,$$

then

$$\frac{1}{n} \leqslant \frac{u_n + v_n}{2}.$$

Since $\sum\limits_{n\geqslant 1} u_n$ *and* $\sum\limits_{n\geqslant 1} v_n$ *converge,* $\sum\limits_{n\geqslant 1} u_n + v_n$ *converges too. According to the comparison theorem* $\sum\limits_{n\geqslant 1} \dfrac{1}{n}$ *converges, which is incorrect.*

Therefore, the convergence of $\sum\limits_{n\geqslant 1} \dfrac{1}{1+n^2 u_n}$ *does not imply the convergence of* $\sum u_n$.

Then $\sum\limits_{n\geqslant 1} u_n$ *diverges.*

Solution to Exercise 9

Let γ be a real number such that $1 < \gamma < \alpha$. Comparison of the growth rates of logarithms and polynomials at infinity shows that:

$$\lim_{n\to\infty} \frac{n^{\gamma}}{n^{\alpha}(\ln n)^{\beta}} = \lim_{n\to\infty} \frac{(\ln n)^{-\beta}}{n^{\alpha-\gamma}} = 0$$

for any value of β. In other words, $u_n = o\left(\frac{1}{n^{\gamma}}\right)$. By comparison with a Riemann series, the series converges.

Now, let's compare it to the series with general term $\frac{1}{n}$. Indeed, we have:

$$\lim_{n\to\infty} \frac{n^{1/n}}{n^{\alpha}(\ln n)^{\beta}} = \lim_{n\to\infty} \frac{1}{n^{1-\alpha}(\ln n)^{\beta}} = +\infty.$$

Therefore, for sufficiently large n, we have $\frac{1}{n} \leqslant u_n$. By the corollary of comparaison theorem, since $\sum_{n\geqslant} \frac{1}{n}$ is divergent series, the series with general term u_n diverges.

If $\beta \leqslant 0$, then $\frac{1}{n} \leqslant u_n$, and we conclude similarly that the series diverges. If $\beta \neq 1$, we have

$$T_n = \int_2^n \frac{dx}{x(\ln x)^{\beta}} = \frac{1}{1-\beta}\left(\frac{1}{(\ln n)^{\beta-1}} - \frac{1}{(\ln 2)^{\beta-1}}\right).$$

If $\beta > 1$, this tends to $\frac{1}{\beta-1}\frac{1}{(\ln(2)^{(\beta-1)}}$, we have $T_n \leqslant \dfrac{1}{\beta-}\dfrac{1}{ln(2)^{\beta-1}}$.

If $\beta < 1$, it is immediately noticeable that (T_n) tends to $+\infty$.

Finally, if $\beta = 1$, we know that

$$\int_2^n \frac{dx}{x(\ln x)} = \ln(\ln n) - \ln(\ln 2),$$

which also tends to $+\infty$.

Next, we consider the case $\beta > 0$. The function $x \mapsto \frac{1}{x(\ln x)^{\beta}}$ is decreasing on $[2, +\infty[$. For $k \geqslant 3$, we have:

$$\int_{k+1}^{k} \frac{dx}{x(\ln x)^{\beta}} \leqslant \frac{1}{k(\ln k)^{\beta}} \leqslant \int_{k}^{k-1} \frac{dx}{x(\ln x)^{\beta}}.$$

Summing these inequalities for k from 3 to n gives:

$$\int_{n+1}^{3} \frac{dx}{x(\ln x)^{\beta}} \leqslant n\sum_{k=3}^{n} \frac{1}{k(\ln k)^{\beta}} \leqslant \int_{n}^{2} \frac{dx}{x(\ln x)^{\beta}}.$$

We conclude for $\alpha = 1$ that if $\beta > 1$, the sequence of partial sums of the series is bounded, and thus, since the series has positive terms, it converges. If $\beta \leqslant 1$, following the same *reasoning as in the previous question, we find that the sequence of partial sums is bounded below by a sequence tending to* $+\infty$*. Therefore, it tends to* $+\infty$ *itself, then series is divergent.*

Chapter 2

The impropers integrals

2.1 Fundamentals of impropers integrals

Definition 2.1.1. *The integral* $\int_a^b f(x)\,dx$ *is an improper integral if:*

1) $a = -\infty$ *or* $b = +\infty$

2) f *is unbounded at one or more points in the interval* $[a, b]$.

3) *Both conditions above are satisfied.*

2.1.1 Integral of the Type $\int_a^{+\infty} f(x)\,dx$ or $\int_{-\infty}^{b} f(x)\,dx$

Definition 2.1.2. *We say the integral* $\int_a^{+\infty} f(x)\,dx$ *(respectively* $\int_{-\infty}^{b} f(x)\,dx$*) converges if* $\lim\limits_{T\to+\infty} \int_a^T f(x)\,dx = l \in \mathbb{R}$ *(respectively if* $\lim\limits_{T\to+\infty} \int_T^b f(x)\,dx = l \in \mathbb{R}$*), and we write:*

$$\int_a^{+\infty} f(x)\,dx = \lim_{T\to+\infty} \int_a^T f(x)\,dx = l \in \mathbb{R}$$

respectively,

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{T\to+\infty} \int_T^b f(x)\,dx = l \in \mathbb{R}.$$

Example 2.1.3. *We want to study the nature of the integral* $\int_0^{+\infty} e^{-x}\,dx$. *For this, we examine the convergence of* $\lim\limits_{T\to+\infty} \int_0^T e^{-x}\,dx$.
We have $\int_0^T e^{-x}\,dx = [-e^{-x}]_0^T = -e^{-T} + 1 \to 1$ *as* $T \to +\infty$, *thus the integral converges and* $\int_0^{+\infty} e^{-x}\,dx = 1$.

Remark 2.1.4.. —

The integral $\int_a^{+T} f(x)\,dx$ *is a Riemann-type integral.*

2.1.2 Riemann Integral of the Type $\int_a^{+\infty} \frac{dx}{x^p}$ with $p \in \mathbb{R}$ and $a > 0$

Definition 2.1.5. *The integral* $\int_a^{+\infty} \frac{dx}{x^p}$ *is called Reimanna integral.*

Proposition 2.1.6. *The Reimann integral* $\int_a^{+\infty} \frac{dx}{x^p}$ *converges if and only if* $p > 1$.
PROOF — *The integral is improper at* $+\infty$. *We then study* $\lim\limits_{T\to+\infty} \int_a^T \frac{dx}{x^p}$. *There are two cases:*

1) *If* $p \neq 1$, *then*

$$\lim_{T\to+\infty} \int_a^T \frac{dx}{x^p} = \lim_{T\to+\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_a^T = \lim_{T\to+\infty} \frac{T^{1-p} - a^{1-p}}{1-p}$$

   This limit converges only if $p > 1$.

2) *If* $p = 1$, *we get*

$$\lim_{T\to+\infty} \int_a^T \frac{dx}{x} = [\ln(x)]_a^T = \lim_{T\to+\infty} \ln(T) - \ln(a) = +\infty,$$

   hence the integral diverges.

Conclusion:

The Riemann integral of the type $\int_a^{+\infty} \frac{dx}{x^p}$ *converges if and only if* $p > 1$.

Studying the convergence of an integral requires the ability to compute the antiderivative, which is not always possible. In such cases, convergence criteria are used.

2.2 Integral of non-negative functions:

Definition: *We say the improper integral of the type* $\int_a^{+\infty} f(x)\,dx$ *(respectively* $\int_{-\infty}^b f(x)\,dx$*) is of non-negative terms if* $f(x) \geqslant 0$ *on* $[a, +\infty[$ *(respectively* $f(x) \geqslant 0$ *on* $]-\infty, b]$*).*

2.2.1 Propositions and Criteria for impropers integrals

Proposition 2.2.1. *The integral $\int_a^{+\infty} f(x)\,dx$ converges if and only if*

$$\exists M > 0, \quad \forall T \in [a, +\infty[, \quad \int_a^T f(x)\,dx < M$$

PROOF — *Let $F(T) = \int_a^T f(x)\,dx$.*

Consider T_1, T_2 such that $T_1 > T_2$. We have

$$F(T_1) - F(T_2) = \int_{T_2}^{T_1} f(x)\,dx > 0$$

If we assume that F is bounded above, then it must converge.

Conversely, if $\int_a^{+\infty} f(x)\,dx$ converges, then $F(T)$ converges, hence it is bounded above. Since f is non-negative, F is increasing, thus it converges.

Theorem 2.2.2.. — ***(Comparison Criterion)***

Let f and g be two non-negative functions on an interval $[a, +\infty[$ with $a \in \mathbb{R}$.
Let $F = \int_a^{+\infty} f(x)\,dx$ and $G = \int_a^{+\infty} g(x)\,dx$.
Suppose $f(x) \leqslant g(x)$ for all $x \in [a, +\infty[$.

- *If G converges, then F converges.*

- *Contrapositively, if F diverges, then G diverges.*

PROOF — *Suppose that G converges. According to the previous proposition, $\int_a^T g(x)\,dx$ is bounded above, i.e.,*

$$\exists M > 0, \quad \forall T \in [a, +\infty[, \quad \int_a^T g(x)\,dx < M$$

By hypothesis, $f(x) \leqslant g(x)$ for all $x \in [a, +\infty[$, hence

$$\forall T \in [a, +\infty[, \quad \int_a^T f(x)\,dx < M$$

According to the previous proposition, $\int_a^{+\infty} f(x)\,dx$ converges.

Example 2.2.3. *1) Consider the improper integral* $\int_1^{+\infty} \frac{dx}{(1+x^2)(4+x^2)}$.

This integral is of non-negative terms, and we have:

$$\frac{1}{(1+x^2)(4+x^2)} \leqslant \frac{1}{x^4}$$

The integral $\int_1^{+\infty} \frac{dx}{x^4}$ *is a convergent Riemann integral, hence* $\int_1^{+\infty} \frac{dx}{(1+x^2)(4+x^2)}$ *converges.*

2) Consider now the improper integral $\int_2^{+\infty} \frac{dx}{(x-1)}$.

This integral is of non-negative terms, and we have:

$$\frac{1}{(x-1)} \geqslant \frac{1}{x}$$

The integral $\int_2^{+\infty} \frac{dx}{x}$ *is a divergent Riemann integral. Therefore,* $\int_2^{+\infty} \frac{dx}{(x-1)}$ *converges.*

Corollary 2.2.1. *Let* f *and* g *be two non-negative functions on an interval* $[a, +\infty[$ *with* $a \in \mathbb{R}$.

Let $F = \int_a^{+\infty} f(x)\,dx$ *and* $G = \int_a^{+\infty} g(x)\,dx$.
Suppose $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = c \in [0, +\infty]$.

1) If $c > 0$, *then integrals* F *and* G *have the same nature.*

2) If $c = 0$ *and* G *converges, then* F *converges.*

3) If $c = +\infty$ *and* G *diverges, then* F *diverges.*

PROOF — *If* $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = c \in [0, \infty]$, *then*

$$\forall \epsilon > 0, \exists \alpha > 0, \forall x \geqslant \alpha, \quad \left| \frac{f(x)}{g(x)} - c \right| < \epsilon$$

Thus,

$$c - \epsilon \leqslant \frac{f(x)}{g(x)} \leqslant c + \epsilon, \quad \forall x \geqslant \alpha$$

Consequently,

$$(c-\epsilon)g(x) \leqslant f(x) \leqslant (c+\epsilon)g(x)$$

1) Suppose $c > 0$: Apply the comparison criterion to the above inequality and conclude that if G converges, then F converges, and if G diverges, then F also diverges.

2) If $c = 0$: We get

$$\forall \epsilon > 0, \exists \alpha > 0, \forall x \geqslant \alpha, \quad f(x) < \epsilon g(x)$$

By applying the comparison criterion, we obtain the result.

3)

Example 2.2.4. *We want to study the nature of the integral $\int_1^{+\infty} \frac{dx}{x^2\sqrt{2+x^2}}$.*

Let $f(x) = \frac{1}{x^2\sqrt{2+x^2}}$. Since $f(x)$ is asymptotically equivalent to $\frac{1}{x^3}$, we have $\lim_{x\to+\infty} \frac{f(x)}{\frac{1}{x^3}} = 1$.

The integral $\int_1^{+\infty} \frac{dx}{x^3}$ is a convergent Riemann integral, hence $\int_1^{+\infty} f(x)\,dx$ converges.

2.2.2 Absolutely convergent generalized integrals

The notion of absolute convergence is useful when dealing with integrals of functions of arbitrary sign.

Definition 2.2.1. *The integral $\int_a^{+\infty} f(x)\,dx$ with $a \in \mathbb{R}$ is said to be absolutely convergent if the integral $\int_a^{+\infty} |f(x)|\,dx$ converges.*

Example 2.2.5. *We examine the absolute convergence of the integral $\int_1^{+\infty} \frac{\sin(x^2)}{x^2}dx$, i.e., the convergence of $\int_1^{+\infty} \left|\frac{\sin(x^2)}{x^2}dx\right|$. To determine this, we use the comparison criterion:*

$$\frac{|\sin(x^2)|}{x^2} \leqslant \frac{1}{x^2}$$

Since $\int_1^{+\infty} \frac{dx}{x^2}$ converges, it follows that $\int_1^{+\infty} \frac{|\sin(x^2)|}{x^2}$ converges, implying the absolute convergence of $\int_1^{+\infty} \frac{\sin(x^2)}{x^2}\,dx$.

Theorem 2.2.6.. — *I*

f the integral $\int_a^{+\infty} f(x)\,dx$ converges absolutely, then it converges, and we have:

$$\left|\int_a^{+\infty} f(x)\,dx\right| \leqslant \int_a^{+\infty} |f(x)|\,dx$$

In other words, absolute convergence implies stronger convergence.

PROOF — *This is a consequence of the Cauchy criterion.*

Assume the integral $\int_a^{+\infty} f(x)\,dx$ *converges absolutely. Then:*

$$\forall \epsilon > 0, \quad \exists M > a, \quad \forall u, v > M, \quad \textit{we have} \quad \left| \int_u^v f(x)\,dx \right| < \epsilon$$

Thus,

$$\int_a^{+\infty} f(x)\,dx \leqslant \int_a^{+\infty} |f(x)|\,dx$$

Therefore, by the Cauchy criterion, the integral $\int_a^{+\infty} f(x)\,dx$ *converges.*

2.2.3 Semi-convergent generalized integral

Definition 2.2.2. *A generalized integral* $\int_a^{+\infty} f(x)\,dx$ *is said to be semi-convergent if it converges but does not converge absolutely.*

Example 2.2.7. *The integral* $\int_1^{+\infty} \frac{\sin(x)}{x}\,dx$ *is semi-convergent. Indeed, convergence will be shown subsequently by the Abel's criterion. Now, we demonstrate that this integral does not converge absolutely.*

We have

$$\frac{\sin(x)}{x} \geqslant \frac{\sin^2(x)}{x} = \frac{1-\cos(2x)}{2x}$$

By integration by parts, we get:

$$\int_1^T \frac{1-\cos(2x)}{2x}\,dx = [\ln(x)]_1^T - \frac{1}{4}\left[\frac{\sin(2x)}{x}\right]_1^T - \frac{1}{4}\int_1^T \frac{\sin(2x)}{x^2}\,dx$$

It can be shown easily by the comparison criterion that the integral $\int_1^T \frac{\sin(2x)}{x^2}\,dx$ *converges absolutely. Also, it can be shown that* $\left[\frac{\sin(2x)}{x}\right]_1^T$ *converges as* $T \to +\infty$*, and since* $[\ln(x)]_1^T \to +\infty$ *as* $T \to +\infty$*, we obtain the result.*

In the following, we present a result that could be used in the case where an integral does not converge absolutely.

Theorem 2.2.8.. — ***Abel's Criterion***

Let $f, g : [a, +\infty[$ *be two locally integrable functions. Assume:*

1) f *is positive, decreasing, and tends to zero at infinity.*

2) There exists a constant $M > 0$ such that

$$\forall T > a, \quad \int_a^T g(x)\,dx < M$$

Then,

$$\int_a^T f(x)g(x)\,dx \text{ converges.}$$

PROOF — *Let $G(T) = \int_a^T g(x)\,dx$. If G is bounded, then there exists $M > 0$ such that $G(x) \leqslant M$. By integration by parts, we have:*

$$\int_a^T f(x)g(x)\,dx = [f(x)G(x)]_a^T - \int_a^T f'(x)g(x)\,dx$$

Since G is bounded and f converges to 0, $[f(x)G(x)]_a^T$ converges to 0 as $T \to +\infty$. Now, we show the absolute convergence of $\int_a^T f'(x)g(x)\,dx$. Since f is decreasing, we have:

$$f'(x)G(x) \leqslant -f'(x)M$$

As $\int_a^T -f'(x)\,dx = -f(T) + f(a)$ and by hypothesis f is positive, decreasing, and tends to 0 at infinity, then the integral $\int_a^T -f'(x)\,dx$ converges, which implies the absolute convergence of $\int_a^T f'(x)g(x)\,dx$, hence the convergence.

Example 2.2.9. *We aim to study the nature of the integral $\int_1^{+\infty} \frac{\sin(x)\,dx}{x}$. Let $f(x) = \frac{1}{x}$ and $g(x) = \sin(x)$ for $x \in [1, +\infty[$.*

- *It is evident that f is positive, decreasing, and tends to 0 near infinity.*
- *$\exists M = 2 > 0$, for all $T > 1$, $\int_1^T \sin(x)\,dx = -\cos(T) + \cos(1) < M$. By Abel's criterion, the integral $\int_1^{+\infty} \frac{\sin(x)\,dx}{x}$ converges.*

2.2.4 Comparison between Improper Integral and Numerical Series

Proposition 2.2.1. *If f is a function defined on an interval $[a, +\infty[$ that is positive and decreasing, then $\int_a^{+\infty} f(x)\,dx$ converges if and only if the series $\sum_{n\geqslant n_0} f(n)$ converges.*

Example 2.2.10. *We want to determine the nature of the integral $\int_2^{+\infty} \frac{dx}{x\ln^2(x)}$. Let's define $f(x) = \frac{1}{x\ln^2(x)}$. This function is continuous, positive, and decreasing on the interval $[2, +\infty[$. Therefore, $\int_2^{+\infty} \frac{dx}{x\ln^2(x)}$ and the series $\sum_2^{+\infty} \frac{1}{n\ln^2(n)}$ share the same nature. Since the series $\sum_2^{+\infty} \frac{1}{n\ln^2(n)}$ converges (it's a Bertrand series), $\int_2^{+\infty} \frac{dx}{x\ln^2(x)}$ also converges.*

2.3 Improper Integral of the form $\int_a^b f(x)\,dx$ with $a, b \in \mathbb{R}$

Now, we are interested in integrals of the form $\int_a^b f(x)\,dx$ *where* $a, b \in \mathbb{R}$ *and* f *is not bounded at one or more points in the interval* $[a, b]$.

In the following, we assume that f *is not bounded at the point* b *in the interval* $[a, b[$, *which means we have an improper integral at* b.

Definition 2.3.1. *Let* $I = \int_a^b f(x)\,dx$ *be an improper integral at* b.

We say that the integral I *converges if*

$$\lim_{t \to b^-} \int_a^t f(x)\,dx = l \in \mathbb{R}$$

and we have

$$\int_a^b f(x)\,dx = l$$

1) *If* $\lim_{t \to b^-} \int_a^t f(x)\,dx$ *diverges, we say that the integral* I *diverges.*

2) *If* f *is not bounded at* a, *then we say that the integral* $I = \int_a^b f(x)\,dx$ *converges if* $\lim_{t \to a^+} \int_t^b f(x)\,dx = l \in \mathbb{R}$.

3) *If* f *is not bounded at both* a *and* b, *then the integral* $\int_a^b f(x)\,dx$ *converges if both* $\int_a^c f(x)\,dx$ *and* $\int_c^b f(x)\,dx$ *converge independently of the choice of* c.

Example 2.3.1. 1) *The integral* $I = \int_0^1 \frac{dx}{x^{2/3}}$ *converges. Indeed, this integral is improper at* 0 *and* $\lim_{t \to 0^+} \int_t^1 \frac{dx}{x^{2/3}} = \lim_{t \to 0^+} [3x^{1/3}]_t^1 = \lim_{t \to 0^+} 3(1 - t^{1/3}) = 3$. *Therefore, the integral* I *converges.*

2) *The integral* $J = \int_0^1 \frac{dx}{\sqrt{1-x}}$ *diverges. Indeed,*

$$\lim_{t \to 1^-} \int_0^t \frac{dx}{1-x} = \lim_{t \to 1^-} [-\ln(1-x)]_0^t = \lim_{t \to 1^-} \ln(1-t) = +\infty.$$

Therefore, the integral J *diverges.*

If f *is not defined at a point* $b \in \mathbb{R}$ *but has a continuous extension at that point, then the integral* $\int_a^b f(x)\,dx$ *converges. Such integrals are called falsely improper integrals.*

2.3.1 Riemann Integral of the Type $\int_0^b \frac{dx}{x^p}$ with $p \in \mathbb{R}$

Proposition 2.3.1. *The integral $\int_0^b \frac{dx}{x^p}$ converges if and only if $p < 1$.*

Proof — *This integral is improper at 0, so according to the definition:*

$$\int_0^b \frac{dx}{x^p} = \lim_{t\to 0^+} \left[\frac{1}{(1-p)x^{p-1}}\right]_t^b = \lim_{t\to 0^+} \frac{1}{(1-p)b^{p-1}} - \frac{1}{(1-p)t^{p-1}} \quad \text{if } p \neq 1.$$

This limit converges if $p < 1$.

If $p = 1$, then we get:

$$\int_0^b \frac{dx}{x^p} = \lim_{t\to 0^+} [\ln(x)]_t^b = \int_0^b \frac{dx}{x} = \lim_{t\to 0^+} \ln(b) - \ln(t) = +\infty.$$

Remark 2.3.2.. —

The integrals $\int_a^b \frac{dx}{(b-x)^p}$ and $\int_a^b \frac{dx}{(x-a)^p}$ converge if and only if $p < 1$.

2.4 Improper Integral of positive functions over bounded intervals

The results for improper integrals of positive functions seen in the previous section hold for improper integrals of positive functions over a bounded interval as well. For the convenience of the reader, let's recall them.

Proposition 2.4.1. *The improper integral $\int_a^b f(x)\,dx$ converges at b if:*

$$\exists M > 0, \quad \forall \epsilon > 0, \quad \text{such that} \quad a < b - \epsilon \quad \text{implies} \quad \int_a^{b-\epsilon} f(x)\,dx < M.$$

Theorem 2.4.1.. — ***Comparison Criterion***

Let f and g be two positive functions on an interval $[a, b[$ such that $\lim_{t\to b} f(x) = \lim_{t\to b} g(x) = +\infty$.

Assume $f(x) \leqslant g(x)$ for all $x \in [a, b[$. Then:

1) If $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ converges.

2) If $\int_a^b f(x)\,dx$ diverges, then $\int_a^b g(x)\,dx$ diverges.

Example 2.4.2. *Let's study the nature of the integral* $\int_1^2 \frac{dx}{\sqrt{x^2-1}}$.

We have:

$$\frac{dx}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x-1}\sqrt{x+1}} \leqslant \frac{1}{\sqrt{x-1}}.$$

Since the integral $\int_1^2 \frac{1}{\sqrt{x-1}}$ *converges,* $\int_1^2 \frac{dx}{\sqrt{x^2-1}}$ *converges.*

Corollary 2.4.1. *Let* f *and* g *be two positive functions on an interval* $[a,b[$ *such that* $\lim_{t\to b} f(x) = \lim_{t\to b} g(x) = +\infty$.

1) *If* $\lim_{t\to b} \frac{f(x)}{g(x)} = c > 0$, *then the integrals* $\int_a^b f(x)\,dx$ *and* $\int_a^b g(x)\,dx$ *have the same nature.*

2) *If* $\lim_{t\to b} \frac{f(x)}{g(x)} = 0$ *and* $\int_a^b g(x)\,dx$ *converges, then* $\int_a^b f(x)\,dx$ *converges.*

3) *If* $\lim_{t\to b} \frac{f(x)}{g(x)} = +\infty$ *and* $\int_a^b g(x)\,dx$ *diverges, then* $\int_a^b f(x)\,dx$ *diverges.*

Example 2.4.3. *Let's study the nature of the integral* $\int_0^1 \frac{dx}{\sin(x)}$. *This integral is improper at* 0.

We have:

$$\lim_{t\to 0} \frac{\frac{1}{\sin(x)}}{\frac{1}{x}} = 1,$$

so $\int_0^1 \frac{dx}{\sin(x)}$ *and* $\int_0^1 \frac{dx}{x}$ *have the same nature. Since* $\int_0^1 \frac{dx}{x}$ *diverges,* $\int_0^1 \frac{dx}{\sin(x)}$ *diverges.*

2.5 Absolute Convergence

Definition 2.5.1. *The improper integral* $\int_a^b f(x)\,dx$ *at a point in the interval* $[a,b]$ *is said to be absolutely convergent if the integral* $\int_a^b |f(x)|\,dx$ *converges.*

Example 2.5.1. *Let's study the absolute convergence of the integral* $\int_0^{2\pi} \frac{\sin(x)\,dx}{\sqrt{x}}$. *This integral is not of positive terms and is improper at* 0.

We have:

$$\forall x \in]0, 2\pi], \qquad \left|\frac{\sin(x)}{\sqrt{x}}\right| \leqslant \frac{1}{\sqrt{x}}.$$

Since the integral $\int_0^{2\pi} \frac{dx}{\sqrt{x}}$ *converges,* $\int_0^{2\pi} \frac{\sin(x)\,dx}{\sqrt{x}}$ *converges absolutely.*

Theorem 2.5.2.. — *I*

f the improper integral $\int_a^b f(x)\,dx$ converges absolutely, then it converges and we have:

$$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx.$$

Remark 2.5.3.. —

An integral that converges without converging absolutely is called semi-convergent.

2.6 Improper Integral at more than one point

Consider the improper integral $\int_a^{+\infty} f(x)\,dx$ such that f is unbounded at a. The convergence of this integral requires the convergence of both $\int_a^b f(x)\,dx$ and $\int_b^{+\infty} f(x)\,dx$ independently of the choice of b. In this case, we have:

$$\int_a^{+\infty} f(x)\,dx = \int_a^b f(x)\,dx + \int_b^{+\infty} f(x)\,dx.$$

2.7 Calculation of improper integral with change of variable

Let $f :]a,b[\to \mathbb{R}$ and $\phi :]\alpha,\beta[\to]a,b[$ be C^1 functions that are bijective. Then the improper integrals $\int_a^b f(x)\,dx$ and $\int_\alpha^\beta f(\phi(x))(\phi)'dx$ have the same nature, and if they converge then:

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\phi(x))(\phi)'dx.$$

Example 2.7.1. *Let's calculate the improper integral $\int_1^{+\infty} \frac{dx}{x(1+\ln^2(x))}$. It is improper at $+\infty$ because the function $t \to f(x) = \frac{1}{x(1+\ln^2(x))}$ is continuous on $[1,+\infty[$. To do this, we consider the change of variable $x = \phi(t) = \ln(t)$ defined from $[1,+\infty[$ to $[0,+\infty[$. ϕ is bijective because it is continuous and strictly increasing on $[1,+\infty[$. Hence,*

$$\int_1^{+\infty} \frac{dx}{x(1+\ln^2(x))} = \int_0^{+\infty} \frac{dx}{1+x^2} = \lim_{u\to+\infty} [\arctan(x)]_0^u = \lim_{u\to+\infty} \arctan(u) = \frac{\pi}{2}.$$

2.8 Integration by parts for improper integrals

Let f and g be C^1 functions on $]a,b[$. If the function fg has finite limits at a and b, then:

$$\int_a^b f(x)g'(x)\,dx = \lim_{t\to b}[f(x)g(x)]_a^t - \lim_{t\to b}\int_a^t f'(x)g(x)\,dx.$$

Example 2.8.1. *Let's calculate the improper integral* $\int_0^{+\infty} te^{-t}\,dt$*. The function* $t \to f(t) = te^{-t}$ *is continuous on* $[0, +\infty[$*, so we have an improper integral at* $+\infty$*.*

We perform integration by parts with $f(x) = x$ *and* $g(t) = e^{-x}$*. For all* $x > 0$*:*

$$\int_0^{+\infty} xe^{-x}\,dx = \lim_{T\to+\infty} \int_0^{T} xe^{-x}\,dx$$

$$= \lim_{t\to+\infty} [-xe^{-x}]_0^x - \int_0^x -e^{-x}\,dx = \lim_{x\to+\infty} (-xe^{-x} - (e^{-x} - 1)) = 1.$$

Therefore, the integral $\int_0^{+\infty} xe^{-x}\,dx$ *converges and we have* $\int_0^{+\infty} xe^{-x}\,dx = 1$*.*

2.9 Exercises:

Exercise 01:

Calculate and determine the nature of the following improper integrals:

$$1)\int_0^{+\infty} \frac{dx}{(1+e^x)(1+e^{-x})},\quad 2)\int_0^{+\infty} \frac{e^{-\sqrt{x}}dx}{x},\quad 3)\int_1^{+\infty} \ln(x)\,dx,\quad 4)\int_0^{1} \ln(x)\,dx.$$

Exercise 02:

Study the nature of the following integrals:

$$1)\int_0^{+\infty} e^{-x^2}\,dx;\quad 2)\int_0^{+\infty} \frac{1+\sin(x)}{1+\sqrt{x^3}}\,dx;\quad 3)\int_1^{+\infty} \frac{\ln(x)}{x^2}\,dx,\quad 4)\int_{-\infty}^{+\infty} \frac{e^{-ax}}{1+e^x}\,dx \quad \textit{where} \quad a\in\mathbb{R},$$

$$5)\int_0^{1} \frac{\arctan(x)}{x^2}\,dx,\quad 6)\int_0^{+\infty} \frac{e^{-x}}{(x^2+1)\sqrt{x}}\,dx.$$

Exercise 03:

The goal of this exercise is to calculate the value of the integral $\int_0^{+\infty} \dfrac{\ln(x)}{1+x^2}\,dx$*.*

1) *Study the nature of the integral* $I = \int_0^{1} \dfrac{\ln(x)}{1+x^2}\,dx$*.*
2) *Using the change of variable* $y = \frac{1}{x}$*, establish a relationship between integral* I *and integral* J *defined by* $J = \int_1^{+\infty} \dfrac{\ln(x)}{1+x^2}\,dx$*, then deduce its nature.*
3) *Deduce the value of the integral* $\int_0^{+\infty} \dfrac{\ln(x)}{1+x^2}\,dx$*.*

Exercise 04:

a) Study the nature of the integral defined by:

$$I = \int_0^{+\infty} \frac{x^3 - 1}{(1 + x^2)\sqrt{1 + x^6}}\, dx.$$

b) Using a change of variable, find the value of I.

Exercise 05:

Consider the integrals I and J defined by: $I = \int_0^{\frac{\pi}{2}} \ln(\sin(x))\, dx$ and $J = \int_0^{\frac{\pi}{2}} \ln(\cos(x))\, dx$.

1) Show that I converges.

2) Show that $I = J$.

3) Determine the values of I and J.

Exercise 06:

Consider the integral I_n defined by:

$$I_n = \int_0^{+\infty} x^n e^{-x}\, dx, \quad n \in \mathbb{N}.$$

1) Calculate I_0 and determine its nature.

2) Using integration by parts, establish a relation between I_n and I_{n-1}, and deduce the nature of I_n.

Exercise07:

For $\alpha, \beta \in \mathbb{R}$, we want to determine the nature of the integral $\int_e^{+\infty} \frac{dx}{x^\alpha (\ln x)^\beta}$.

1) Suppose $\alpha > 1$. By comparing with a Riemann integral, demonstrate that the integral in question converges.

2) Suppose $\alpha = 1$. Compute $\int_e^T \frac{dx}{x(\ln x)^\beta}$ for $\beta < 1$, and determine the values of β for which the integral converges.

3) Suppose $\alpha < 1$. Show that the integral in question diverges.

2.10 Solution:

Solution of Exercise 1

1) The function $x \to \dfrac{1}{(1+e^x)(1+e^{-x})}$ *is continuous positive on* $[0,+\infty[$*, so we have an improper integral at* $+\infty$*.*

$$\frac{1}{(1+e^x)(1+e^{-x})} = \frac{1}{(1+e^x)(1+e^{-x})} = \frac{1}{(1+e^x)(1+\dfrac{1}{e^x})} = \frac{e^x}{(1+e^x)^2}.$$

We notice it has the form $\dfrac{u'}{(1+u^2)^2}$ *which has the antiderivative* $-\dfrac{1}{1+u}$*, hence:*

$$\int_0^{+\infty} \frac{dx}{(1+e^x)(1+e^{-x})} = \lim_{T\to+\infty} \frac{dx}{(1+e^x)(1+e^{-x})} = \lim_{T\to+\infty} [-\frac{1}{1+e^x}]_0^T = \lim_{T\to+\infty} (\frac{1}{2} - \frac{1}{1+e^T}) = \frac{1}{2}.$$

Therefore, the integral $\int_0^{+\infty} \dfrac{dx}{(1+e^x)(1+e^{-x})}$ *converges.*

2) The function $x \to f(x) = \dfrac{e^{-\sqrt{x}}}{x}$ *is continuous on the interval* $[0,+\infty[$*, so we have an improper integral at* $+\infty$*.*

$$\int_0^{+\infty} \frac{e^{-\sqrt{x}}dx}{x} = -2 \lim_{T\to+\infty} \int_0^T \frac{e^{-\sqrt{x}}dx}{x} = \lim_{T\to+\infty} [-2e^{-\sqrt{x}}]_0^T = \lim_{T\to+\infty} (2 - 2e^{\sqrt{T}}) = 2.$$

Thus, the integral $\int_0^{+\infty} \dfrac{e^{-\sqrt{x}}dx}{x}$ *converges.*

3) The function $x \to \ln(x)$ *is continuous on the interval* $[1,+\infty[$*, so we have an improper integral at* ∞*.*

$$\int_1^{+\infty} \ln(x)dx = \lim_{T\to+\infty} \int_1^T \ln(x)dx = \lim_{T\to+\infty} [x\ln(x) - x]_1^T = \lim_{T\to+\infty} T\ln(T) - T + 1 = +\infty.$$

Therefore, the integral $\int_1^{+\infty} \ln(x)dx$ *diverges.*

4) The function $x \to \ln(x)$ *is continuous on the interval* $]0,1]$*, so we have an improper integral at* 0*.*

$$\int_0^1 \ln(x)dx = \lim_{t\to0^+} \int_t^1 \ln(x)dx = \lim_{t\to0^+} [x\ln(x) - x]_t^1 = \lim_{t\to0^+} (-1 - t\ln(t) + t) = -1.$$

Thus, $\int_0^1 \ln(x)dx$ *converges.*

Solution of Exercise 2

1) The function $x \to e^{-x^2}$ *is continuous on the interval* $[0, +\infty[$, *so we have an improper integral at* $+\infty$.

$$\lim_{x\to+\infty} \frac{e^{-x^2}}{\frac{1}{x^2}} = 0 \quad \textit{and} \quad \int_1^{+\infty} \frac{dx}{x^2}$$

converges, hence by the comparison criterion corollary, $\int_0^{+\infty} e^{-x^2} dx$ *converges.*

2) The function $x \to \dfrac{1+\sin(x)}{1+\sqrt{x^3}}$ *is continuous on the interval* $[0, +\infty[$, *so we have an improper integral at* $+\infty$.

$$\frac{1+\sin(x)}{1+\sqrt{x^3}} \leqslant \frac{2}{x^3} \quad \textit{and} \quad \int_1^{+\infty} \frac{dx}{x^3} \textit{ converges, thus } \int_1^{+\infty} \frac{1+\sin(x)}{1+\sqrt{x^3}} dx$$

converges, hence $\int_0^{+\infty} \dfrac{1+\sin(x)}{1+\sqrt{x^3}} dx$ *converges.*

3) The function $x \to \dfrac{\ln(x)}{x^3}$ *is continuous and positive on the interval* $[1, +\infty[$, *so we have an improper integral at* $+\infty$. $\dfrac{\ln(x)}{x^3} \leqslant \dfrac{1}{x^2}$ *and* $\int_1^{+\infty} \dfrac{1}{x^2} dx$ *is a convergent Riemann integral, hence* $\int_1^{+\infty} \dfrac{\ln(x)}{x^3} dx$ *also converges.*

4) The function $x \to \dfrac{e^{-ax}}{1+e^x}$ *is continuous and positive on the interval* $[-\infty, +\infty[$, *so we have improper integrals at* $-\infty$ *and* $+\infty$.

$$\int_0^{+\infty} \frac{e^{-ax}}{1+e^x} dx \quad \textit{and} \quad \int_{-\infty}^0 \frac{e^{-ax}}{1+e^x} dx.$$

For $x > 0$*:*

$$\frac{e^{-ax}}{1+e^x} \sim \frac{e^{-ax}}{e^x} = e^{-(a+1)x}.$$

$$\int_0^{+\infty} e^{-(a+1)x} = \lim_{T\to+\infty} \int_0^T e^{-(a+1)x} dx = \lim_{T\to+\infty} \left[\frac{e^{-(a+1)x}}{-(a+1)} \right]_0^T = \lim_{T\to+\infty} \frac{1-e^{-(a+1)T}}{a+1}.$$

This limit converges if $a > -1$, *so* $\int_0^{+\infty} \dfrac{e^{-ax}}{1+e^x} dx$ *converges if* $a > -1$.

For $x < 0$*:*

$$\frac{e^{-ax}}{1+e^x} \sim e^{-ax}.$$

$$\int_{-\infty}^0 e^{-ax} = \lim_{T\to-\infty} \int_T^0 e^{-ax} dx = \lim_{T\to-\infty} \left[\frac{e^{-ax}}{a} \right]_T^0 = \lim_{T\to+\infty} \frac{1-e^{-aT}}{-a}.$$

This limit converges if $a < 0$*, so* $\int_{-\infty}^{0} \frac{e^{-ax}}{1+e^x}dx$ *converges if* $a < 0$.

Therefore, $\int_{-\infty}^{+\infty} \frac{e^{-ax}}{1+e^x}dx$ *converges if* $a \in]-1,0[$.

5) The function $x \to \frac{\arctan(x)}{x^2}$ *is defined and continuous on* $]0,1]$*, so we have an improper integral at* 0.

$f(x) \sim \frac{1}{x}$ *and* $\displaystyle\int_0^1 \frac{1}{x}dx$ *diverges (Riemann integral with* $p = 1$*), hence* $\displaystyle\int_0^1 \frac{\arctan(x)}{x^2}dx$ *diverges.*

6) $x \to f(x) = \frac{e^{-x}}{(x^2+1)\sqrt{x}}$ *is defined and continuous on* $]0,+\infty[$*, so we have improper integrals at* 0 *and* $+\infty$.

We examine $\int_0^1 \frac{e^{-x}}{(x^2+1)\sqrt{x}}dx$ *and* $\int_1^{+\infty} \frac{e^{-x}}{(x^2+1)\sqrt{x}}dx$.

At $x = 0$, $f(x) \sim \frac{1}{\sqrt{x}}$ *and* $\int_0^1 \frac{1}{\sqrt{x}}dx$ *converges (Riemann integral with* $p < 1$*), thus* $\int_0^1 f(x)dx$ *converges.*

At $+\infty$, $f(x) \sim \frac{e^{-x}}{x^{5/2}}$ *and* $\dfrac{\frac{e^{-x}}{x^{5/2}}}{\frac{1}{x^{5/2}}} \to 0$ *as* $x \to +\infty$*. Also,* $\int_1^{+\infty} \frac{1}{x^{5/2}}dx$ *converges, so* $\int_1^{+\infty} f(x)dx$ *converges, thus* $\int_0^{+\infty} f(x)dx$ *converges.*

Solution of Exercise 3

1) *The integral* $I = \int_0^1 \frac{ln(x)}{1+x^2}dx$ *is an improper integral at 0.*

   Since $\frac{ln(x)}{1+x^2} \sim ln(x)$ *as* $x \to 0$*, from exercise 1 we know* $\int_0^1 ln(x)dx$ *converges.*

   Therefore, $\int_0^1 \frac{ln(x)}{1+x^2}dx$ *also converges.*

2) *Let* $y = \frac{1}{x}$*, hence* $dx = \frac{-dy}{y^2}$.

   As $x \to 0^+$, $y \to +\infty$*, and as* $x \to \infty$, $y \to 0$*. Thus, we have:*

$$I = \int_{+\infty}^{1} \frac{ln(1/y)}{1+1/y^2}\frac{-dy}{y^2} = \int_1^{+\infty} \frac{ln(y)}{\frac{y^2+1}{y^2}}\frac{-dy}{y^2} = -\int_1^{+\infty} \frac{ln(y)}{1+y^2}dy = -J.$$

   Therefore, J *converges.*

3) As I and J converge we have $\int_0^{+\infty} \frac{ln(x)}{1+x^2}dx = I + J = I - I = 0.$

Solution of Exercise 4

a) We examine the convergence of the integral $I = \int_0^{+\infty} \frac{x^3-1}{(1+x^2)\sqrt{1+x^6}}dx.$

$x \to f(x) = \frac{x^3-1}{(1+x^2)\sqrt{1+x^6}}$ *is defined and continuous on* $[0, +\infty[$.
We have an improper integral at $+\infty$, $f(x)$ *is positive for sufficiently large* x.
$f(x) \sim \frac{1}{x^2}$ *and* $\int_1^{+\infty} \frac{1}{x^2}$ *converges, hence* $\int_1^{+\infty} f(x)dx$ *converges. Also,* $\int_0^1 f(x)dx$ *is a finite integral, so* $\int_0^{+\infty} f(x)dx$ *converges.*

b) Let $y = \frac{1}{x}$, *hence* $dx = -\frac{1}{y^2}dy$, $y \to 0$ *as* $x \to +\infty$, *and* $y \to +\infty$ *as* $x \to 0$.
Substitute into I*:*

$$I = \int_{+\infty}^{0} \frac{\frac{1}{y^3}-1}{(y^2+1)\sqrt{1+1/y^6}}\left(-\frac{1}{y^2}\right)dy = \int_0^{+\infty} \frac{\frac{1}{y^3}-1}{(y^2+1)\sqrt{1+1/y^6}}dy$$

$$= \int_0^{+\infty} \frac{1-y^3}{(1+y^2)\sqrt{1+y^6}}dy$$

$$= -\int_0^{+\infty} \frac{y^3-1}{(1+y^2)\sqrt{1+y^6}}dy = -I.$$

Therefore, $I = -I$, *which implies* $I = 0$.

Solution of Exercise 5 *1) The function* $x \to \ln(\sin(x))$ *is continuous on* $\left]0, \frac{\pi}{2}\right]$ *and* I *is an improper integral at 0.*

Since $\sin(x) \sim x$ *near 0, we have* $\ln(\sin(x)) \sim \ln(x)$*. According to Exercise 1,* $\int_0^{\frac{\pi}{2}} \ln(x)\, dx$ *converges, hence* I *converges.*

2) Let's substitute $x = \frac{\pi}{2} - u$*,* $dx = -du$*. Then:*

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin(\frac{\pi}{2} - u))\,(-du) = \int_0^{\frac{\pi}{2}} \ln(\sin(\frac{\pi}{2} - u))\, du = \int_0^{\frac{\pi}{2}} \ln(\cos(u))\, du = J$$

3)

$$\begin{aligned}
2I = I + J &= \int_0^{\frac{\pi}{2}} \ln(\cos(x))\, dx + \int_0^{\frac{\pi}{2}} \ln(\sin(x))\, dx = \int_0^{\frac{\pi}{2}} \ln(\sin(x)\cos(x))\, dx \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\frac{\sin(2x)}{2}\right) dx = -\frac{\pi \ln 2}{2} + \int_0^{\frac{\pi}{2}} \ln(\sin(2x))\, dx \\
&= -\frac{\pi \ln 2}{2} + \frac{1}{2}\int_0^{\pi} \ln(\sin(t))\, dt = -\frac{\pi \ln 2}{2} + \frac{I}{2} - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \ln(\sin(t))\, dt \\
&= -\frac{\pi \ln 2}{2} + \frac{I}{2} - \frac{1}{2}\int_{\frac{\pi}{2}}^{0} \ln(\sin(\pi - s))\, ds \\
&= -\frac{\pi \ln 2}{2} + \frac{I}{2} - \frac{1}{2}\int_{\frac{\pi}{2}}^{0} \ln(\sin(s))\, ds \\
&= -\frac{\pi \ln 2}{2} + I.
\end{aligned}$$

Thus, $I = J = -\frac{\pi \ln 2}{2}$*.*

Solution of Exercise 6

1)

$$I_0 = \int_0^{+\infty} e^{-x}\, dx = \lim_{T \to +\infty} \left[-e^{-x}\right]_0^T = \lim_{T \to +\infty} (1 - e^{-T}) = 1$$

Thus, I_0 *converges.*

2)

$$\begin{aligned}
I_n = \lim_{T \to +\infty} \int_0^T x^n e^{-x}\, dx &= \lim_{T \to +\infty} \left[-x^n e^{-x}\right]_0^T + \int_0^T n x^{n-1} e^{-x}\, dx \\
&= n \lim_{T \to +\infty} \int_0^T x^{n-1} e^{-x}\, dx = n I_{n-1}, \quad \text{for } n \geqslant 1.
\end{aligned}$$

Therefore, $I_n = nI_{n-1} = n(n-1)I_{n-2} = \ldots = n!$ *for* $n \in \mathbb{N}$*.*

Thus, I_n *converges.*

Solution of Exercise 7 *$t \to f(t) = t^{\gamma} t^{\alpha} (\ln t)^{\beta}$ is defined, continious on $[2, +\infty[$, then we have an improper integral at +*

Let $\gamma \in (1, \alpha)$. Then, we have

$$t^{\gamma} t^{\alpha} (\ln t)^{\beta} = \frac{1}{t^{\alpha-\gamma} (\ln t)^{\beta}} \to 0$$

and thus, denoting f as the function, we have $f(t) = o\left(\frac{1}{t^{\gamma}}\right)$. Since $\int_e^{+\infty} \frac{dt}{t^{\gamma}}$ converges, the same holds true for $\int_e^{+\infty} f$.

If $\alpha = 1$, then the function is of the form $\frac{u'}{u^{\beta}}$. Therefore, it admits a primitive of the form $\frac{1}{1-\beta} \frac{1}{u^{1-\beta}}$ if $\beta \neq 1$, and $\ln |\ln u|$ if $\beta = 1$. For $\beta \neq 1$, we have

$$\int_X^e \frac{dt}{t(\ln t)^{\beta}} = \left[\frac{1}{1-\beta} \frac{1}{(\ln t)^{1-\beta}} \right]_e^X = \frac{1}{1-\beta} \left[(\ln X)^{1-\beta} - (\ln e)^{1-\beta} \right].$$

As X tends to $+\infty$, this converges to a finite limit if and only if $\beta > 1$. In the case where $\beta = 1$, the primitive is calculated slightly differently:

$$\int_X^e \frac{dt}{t \ln t} = [\ln |\ln t|]_e^X = \ln |\ln X| - \ln |\ln e| = \ln \ln X.$$

This tends to $+\infty$, hence the integral does not converge.

It is observed that $\frac{1}{t} f(t) = \frac{t^{\alpha-1}}{(\ln t)^{\beta}} \to 0$, which means $\frac{1}{t} = o(f(t))$.

Since $\int_e^{+\infty} \frac{dt}{t}$ diverges, the same is true for $\int_e^{+\infty} f(t)\, dt$.

In conclusion, the studied integral converges if and only if $\alpha > 1$ or $\alpha = 1$ and $\beta > 1$.

Chapter 3

Ordinary differential equations

3.1 Fundamentals of differential equations

Definition 3.1.1. *An ordinary differential equation of the order is a relation of the form* $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$*, where:*

1) y*: function of* x*,*

2) y'*: derivative of* y*,*

3) f*: continuous function.*

An ordinary differential equation is often denoted as O.D.E.

To solve an ordinary differential equation, one must find the function y *that satisfies it, which is analytically possible only in certain cases of first and second orders.*

3.2 First Order Ordinary Differential Equations:

Separable Variables Equations:

Definition 3.2.1. *Let* $f : I \to \mathbb{R}$ *and* $g : I \to \mathbb{R}$ *be two continuous functions. The differential equation* $y'g(y) = f(x)$ *is called an O.D.E. with separable variables.*

Resolution Method:

$$y'g(y) = f(x) \Rightarrow \frac{dy}{dx}g(y) = f(x) \Rightarrow g(y)\,dy = f(x)\,dx \Rightarrow \int g(y)\,dy = \int f(x)\,dx$$

$$\Rightarrow G(y) + C_1 = F(x) + C_2$$

where F is the antiderivative of f, G is the antiderivative of g, and $C_1, C_2 \in \mathbb{R}$.

Example 3.2.1.

$$3y'(x^2-1) - 2xy = 0 \Rightarrow 3y'(x^2-1) = 2xy \Rightarrow \frac{3}{y}\frac{dy}{dx} = \frac{2x}{x^2-1}$$

$$\Rightarrow \frac{dy}{y} = \frac{2x\,dx}{3(x^2-1)} \Rightarrow \ln(|y|) = \frac{1}{3}\ln(|x^2-1|) + C \quad C \in \mathbb{R}$$

$$\Rightarrow |y| = e^C|x^2-1|^{1/3} \Rightarrow y = k(x^2-1)^{1/3} \quad \textit{where} \quad k = \pm e^C \in \mathbb{R}^*$$

It is noted that the solution $y = 0$ is also a solution, thus the general solution is $y = k(x^2-1)^{1/3}$.

3.2.1 Homogeneous Differential Equations:

Definition 3.2.2. *An ordinary differential equation of the form $y' = f\left(\frac{y}{x}\right)$ is called a homogeneous equation.*

Solution Method:

Let $t = \frac{y}{x} \Rightarrow y = tx$, hence $y' = t'x + t$.

Substituting into the O.D.E. $y' = f\left(\frac{y}{x}\right)$, we obtain:

$$t'x + t = f(t) \Rightarrow t'x = f(t) - t \Rightarrow \frac{t'}{f(t)-t} = \frac{1}{x}$$

This gives us a separable variables O.D.E.

Example 3.2.2. *Consider the O.D.E $x^2y' = x^2 + xy - y^2$, $y(1) = 0$. By setting $z = \dfrac{y}{x}$, such that $y' = xz' + z$, the equation becomes:*

$$x^3z' = x^2(1-z^2).$$

This equation is rewritten as

$$\frac{z'}{1-z^2} = \frac{1}{x},$$

then after partial fraction decomposition,

$$\frac{z'}{1-z} + \frac{z'}{1+z} = \frac{2}{x}.$$

Integrating yields the existence of $C \in R$ *such that*

$$ln \mid \frac{1+z}{1-z} \mid = 2ln \mid \ x \mid \ +C.$$

Setting $C = ln\lambda$ *with* $\lambda > 0$*, this further simplies to*

$$ln \mid \frac{1+z}{1-z} \mid = ln \mid \ \lambda x^2 \mid$$

Solving this equation gives

$$y = x(\mid \ \frac{\lambda x^2 - 1}{\lambda x^2 + 1} \mid).$$

The condition $y(1) = 0$ *determines* $\lambda = 1$*, thus the solution of the equation is the function*

$$y(x) = x(\mid \ \frac{x^2 - 1}{x^2 + 1} \mid, \quad x \in I\!R.$$

3.2.2 Linear Differential Equations:

Definition 3.2.3. *Let* $f : I \to \mathbb{R}$ *and* $a : I \to \mathbb{R}$ *be continuous functions. The differential equation of the form* $y' + a(x)y = f(x)$ *is called a linear O.D.E.*

1) *If* $f(x) = 0$*, it becomes a homogeneous linear O.D.E.*

2) *If* $f(x) \neq 0$*, it is a non-homogeneous linear O.D.E.*

3.2.3 Solution Method for Homogeneous Linear O.D.E.:

$$y' + a(x)y = 0 \Rightarrow y' = -a(x)y \Rightarrow \frac{y'}{y} = -a(x)$$

This results in a separable variables O.D.E., and the general solution is:

$$y = ke^{-\int a(x)\,dx}, \quad k \in \mathbb{R}$$

Example 3.2.3.

$$y' + xy = 0 \Rightarrow y' = -xy \Rightarrow \frac{y'}{y} = -x \Rightarrow \frac{dy}{y} = -x\,dx$$

Integrating both sides:

$$\ln(|y|) = -\frac{x^2}{2} + c, \quad c \in \mathbb{R}$$

$$=> |y| = e^c e^{-\frac{x^2}{2}}$$

$$=> y = ke^{-\frac{x^2}{2}}, \quad k \in \mathbb{R}^*$$

Note that $y = 0$ *is also a solution, hence the general solution is* $y = ke^{-\frac{x^2}{2}}, \quad k \in \mathbb{R}.$

3.2.4 Solution Method for Non-Homogeneous Linear O.D.E.:

Proposition 3.2.1. *The general solution of the non-homogeneous linear O.D.E.* $y'+a(x)y = f(x)$ *is the sum of the general solution of the homogeneous linear O.D.E.* $y' + a(x)y = 0$ *with a particular solution of the non-homogeneous O.D.E.* $y' + a(x)y = f(x)$.

Proof — *Let* (1) *denote the equation* $y' + a(x)y = b(x)$ *and* (2) *denote the equation* $y' + a(x)y = 0$.

Let f_1 *be the solution of equation* (1), f_2 *the general solution of equation* (2), *and* f_p *a particular solution of equation* (1). *We show that* $f_2 + f_p$ *is a solution of* (1):

$$f_2'(x)+f_p'(x)+a(x)(f_2(x)+f_p(x)) = f_2'(x)+a(x)f_2(x)+f_p'(x)+a(x)f_p(x) = 0+b(x) = b(x).$$

Example 3.2.4. *Consider the following O.D.E.:*

$$y' + xy = x \quad \textit{(1)}$$

To solve this, we consider the corresponding homogeneous O.D.E.:

$$y' + xy = 0 \quad \textit{(2)}$$

Equation (2) was solved in the previous example, and its general solution is:

$$y = ke^{-\frac{x^2}{2}}, \quad k \in \mathbb{R}$$

Notice that $y_p = 1$ *is a particular solution of O.D.E. (1). Therefore, its general solution is:*

$$y = ke^{-\frac{x^2}{2}} + 1, \quad k \in \mathbb{R}$$

Remark 3.2.5. *It is not always straightforward to find a particular solution to a non-homogeneous linear O.D.E. For this, a method called variation of parameters is used, which allows us to find the general solution directly.*

3.2.5 Method of Variation of Parameters:

If $y = ke^{-\int a(x)\,dx}$, $k \in \mathbb{R}$ *is the general solution of the homogeneous O.D.E.* $y' + a(x)y = 0$, *then the general solution of the non-homogeneous O.D.E.* $y' + a(x)y = b(x)$ *is* $y = k(x)e^{-\int a(x)\,dx}$, $k \in \mathbb{R}$, *where* k *is a function of* x *to be determined as follows:*

$$y' = k'(x)e^{-\int a(x)\,dx} - k(x)a(x)e^{-\int a(x)\,dx}$$

Substituting into the non-homogeneous O.D.E. $y' + a(x)y = b(x) \quad (1)$*, we get:*

$$k'(x)e^{-\int a(x)\,dx} - k(x)a(x)e^{-\int a(x)\,dx} + a(x)k(x)e^{-\int a(x)\,dx} = b(x)$$

Thus,

$$k'(x)e^{-\int a(x)\,dx} = b(x)$$

Therefore,

$$k'(x) = b(x)e^{\int a(x)\,dx}$$

To find $k(x)$*, integrate* k'*.*

Example 3.2.6. *We want to solve the non-homogeneous linear O.D.E.* $2xy' - 3y = \sqrt{x} \quad (1)$*. First, we solve the homogeneous O.D.E.* $2xy' - 3y = 0 \quad (2)$*. We have:*

$$2xy' = 3y \Rightarrow 2x\frac{dy}{y} = dx \Rightarrow \frac{dy}{y} = \frac{3}{2}\frac{dx}{x} \Rightarrow \ln(|y|) = \frac{3}{2}\ln(|x|) + c \quad c \in \mathbb{R}$$

So,

$$y = k\sqrt{x^3}, \quad k \in \mathbb{R}^*$$

We observe that $y = 0$ *is also a solution of (2), so its general solution is*

$$y = k\sqrt{x^3}, \quad k \in \mathbb{R}$$

We now seek the general solution of (1) of the form

$$y = k(x)\sqrt{x^3}$$

we now determine the function k*, for which we have:*

$$y' = k'(x)\sqrt{x^3} + \frac{3}{2}k(x)\sqrt{x}$$

Substituting into equation (1):

$$2x\left[k'(x)\sqrt{x^3}\right] = \sqrt{x}$$

Therefore,

$$k'(x) = \frac{1}{2x^2}$$

Thus,

$$k(x) = -\frac{1}{2x} + c, \quad c \in \mathbb{R}$$

Hence, the general solution of equation (1) is:

$$y = -\frac{1}{2}\sqrt{x} + c\sqrt{x^3}, \quad c \in \mathbb{R}$$

3.3 Bernoulli Differential Equation:

Definition 3.3.1. *Let* $a, b : I \to \mathbb{R}$ *be two continuous functions. An equation of Bernoulli is a differential equation of the form* $y' + a(x)y = b(x)y^k \quad \text{with} \quad k \neq 0, 1.$

Solution Method:

Divide the equation by y^k*:*

$$y'y^{-k} + a(x)y^{1-k} = b(x)$$

Let $z = y^{1-k}$*, then* $z' = (1-k)y^{-k}y'$*. The Bernoulli equation transforms into a non-homogeneous linear equation:*

$$\frac{1}{1-k}z' + a(x)z = b(x)$$

Example 3.3.1. *We want to solve the following Bernoulli differential equation:*

$$y' - \frac{4}{x}y = x\sqrt{y} \quad (1)$$

Dividing by $\sqrt{y}$ *gives:*

$$y'y^{-1/2} - \frac{4}{x}y^{1/2} = x$$

Let $z = \sqrt{y}$*, then* $z' = \frac{1}{2}\frac{y'}{\sqrt{y}}$*. The equation (1) transforms into the following non-homogeneous linear equation:*

$$2z' - \frac{4}{x}z = x \quad (2)$$

First, solve the homogeneous equation:

$$2z' - \frac{4}{x}z = 0 \quad (3)$$

The general solution of equation (3) is:

$$z = kx^2, \quad k \in \mathbb{R}$$

To find the general solution of equation (2) using the method of variation of parameters, denote the solution as:

$$z = k(x)x^2, \quad k \in \mathbb{R}$$

where k *is a function to determine.*

We have:

$$z' = k'(x)x^2 + 2k(x)x$$

Substituting into equation (2):

$$2k'(x)x^2 + 4k(x)x - 4k(x)x = x$$

Thus,

$$k'(x) = \frac{1}{2x}$$

Therefore,

$$k(x) = \frac{1}{2}\ln(|x|) + c, \quad c \in \mathbb{R}$$

So, the general solution of equation (2) is:

$$z = x^2 \ln(\sqrt{|x|}) + cx^2, \quad c \in \mathbb{R}$$

Hence, the general solution of equation (1) is:

$$y = z^2 = \left(x^2 \ln(\sqrt{|x|}) + cx^2\right)^2, \quad c \in \mathbb{R}$$

3.3.1 Riccati Differential Equations:

Definition 3.3.2. *Let $a, b, c : I \to \mathbb{R}$ be continuous functions. A Riccati equation is a differential equation of the form $y' + a(x)y = b(x)y^2 + c(x)$.*

Solution Method:

The solution of a Riccati equation is possible if we already have a particular solution y_p. In this case, we can set the variable change $z = y - y_p$, thus $y = z + y_p$ and $y' = z' + y_p'$. Substituting into the Riccati equation, we get:

$$(z' + y_p') + a(x)(z + y_p) = b(x)(z^2 + 2zy_p + y_p^2) + c(x)$$

Since y_p is a solution, we obtain a Bernoulli equation:

$$z' + (a(x) - 2b(x)y_p)z = b(x)z^2$$

Example 3.3.2. *Let's solve the Riccati equation defined by:*

$$(x^2+1)y' = y^2 - 1 \quad \textit{(1)}$$

Notice that $y = 1$ *is one of its particular solutions. Let* $z = y - 1$*, so* $z' = y'$*. Substituting into equation (1):*

$$(x^2+1)z' = ((z+1)^2 - 1)$$

This simplifies to the following Bernoulli equation:

$$(x^2+1)z' - 2z = z^2 \quad \textit{(2)}$$

Divide equation (2) by z^2*:*

$$(x^2+1)\frac{z'}{z^2} - \frac{2}{z} = 1$$

Let $t = \frac{1}{z}$*, then* $t' = -\frac{z'}{z^2}$*. This leads to:*

$$-(x^2+1)t' - 2t = 1 \quad \textit{(3)}$$

Equation (3) is a non-homogeneous linear differential equation. Consider the homogeneous equation:

$$-(x^2+1)t' - 2t = 0 \quad \textit{(4)}$$

Its general solution is:

$$t = ke^{-2\arctan(x)}, \quad k \in \mathbb{R}$$

By varying the constant k*, we find the general solution of equation (3):*

$$t = (-\arctan(x) + c)e^{-2\arctan(x)}, \quad c \in \mathbb{R}$$

Thus, the general solution of equation (2) is:

$$z = \frac{1}{-\arctan(x) + c}e^{-2\arctan(x)}$$

Therefore, the general solution of equation (1) is:

$$y = \frac{1}{-\arctan(x) + c}e^{-2\arctan(x)} + 1$$

3.4 Second-Order Linear Ordinary Differential Equations with Constant Coefficients:

Definition 3.4.1. *Let $a, b \in \mathbb{R}$ and $f : I \to \mathbb{R}$ be a function. A second-order linear ordinary differential equation with constant coefficients is of the form:*

$$y'' + ay' + by = f(x)$$

- *If $f(x) = 0$, it is the homogeneous case.*

- *If $f(x) \neq 0$, it is the non-homogeneous case.*

Remark 3.4.1.. —

If y_1 and y_2 are two solutions of the differential equation $y'' + ay' + by = f(x)$, then $\alpha y_1 + \beta y_2$ is also a solution.

Solution Method:

1) Homogeneous Case Solution Method:

We consider the following second-order equation:

$$r^2 + ar + b = 0$$

We calculate the discriminant $\Delta = a^2 - 4b$.
1) If $\Delta > 0$, the roots of the polynomial equation are real and distinct, r_1 and r_2. The particular solutions of the homogeneous equation $y'' + ay' + by = 0$ are $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$, and the general solution is:

$$y = \alpha e^{r_1 x} + \beta e^{r_2 x}, \quad \alpha, \beta \in I\!R$$

Example 3.4.2. *We want to solve the ODE $y'' - y - 2y = 0$. The characteristic polynomial is $r^2 - r - 2 = 0$ with $\Delta = 9$. The roots are $r_1 = -2$ and $r_2 = 1$. Thus, the particular solutions are $y_1 = e^{-2x}$ and $y_2 = e^{x}$, and the general solution is:*

$$y = \alpha e^{-2x} + \beta e^{x}, \quad \alpha, \beta \in I\!R$$

2) If $\Delta = 0$, there is a double root $r_0 = -\frac{a}{2}$. The particular solutions of the homogeneous equation are $y_1 = e^{r_0 x}$ and $y_2 = xe^{r_0 x}$, and the general solution is:

$$y = \alpha e^{r_0 x} + \beta x e^{r_0 x}, \quad \alpha, \beta \in I\!R$$

Example 3.4.3. *We want to solve the ODE $y'' - 4y + 4y = 0$. The characteristic polynomial is $r^2 - 4r + 4 = 0$ with $\Delta = 0$. The double root is $r_0 = 2$. Thus, the particular solutions are $y_1 = e^{2x}$ and $y_2 = xe^{2x}$, and the general solution is:*

$$y = \alpha e^{2x} + \beta x e^{2x}, \quad \alpha, \beta \in I\!R$$

3) If $\Delta < 0$, the roots are complex $r_1 = -\alpha + i\beta$ and $r_2 = -\alpha - i\beta$, where $\alpha = \frac{a}{2}$ and $\beta = \frac{\sqrt{|\Delta|}}{2}$. The particular solutions of the homogeneous equation are:

$$y = C_1 e^{-\alpha x}\cos(\beta x) + C_2 e^{-\alpha x}\sin(\beta x), \quad C_1, C_2 \in I\!R$$

Example 3.4.4. *We want to solve the ODE $y'' + 2y + 5y = 0$. The characteristic polynomial is $r^2 + 2r + 5 = 0$ with $\Delta = -16$. The complex roots are $r_1 = -1 - 2i$ and $r_2 = -1 + 2i$. Thus, the general solution is:*

$$y = C_1 e^{-x}\cos(2x) + C_2 e^{-x}\sin(2x), \quad C_1, C_2 \in I\!R$$

2) Non-Homogeneous Case Solution Method:

The general solution of the non-homogeneous ODE $y'' + ay' + by = f(x)$ is the sum of a particular solution of the non-homogeneous equation with the general solution of the homogeneous equation $y'' + ay' + by = 0$.

Remark 3.4.5.. —

There are different methods to find a particular solution of a non-homogeneous ODE $y'' + ay' + by = f(x)$, which will be detailed subsequently. Generally, the method of variation of parameters can be used.

Variation of Parameters:

If $y = c_1 y_1 + c_2 y_2$ is the general solution of the homogeneous equation $y'' + ay' + by = 0$, then the general solution of the non-homogeneous equation $y'' + ay' + by = f(x)$ is:

$$y = c_1(x) y_1 + c_2(x) y_2,$$

where $c_1(x)$ and $c_2(x)$ are functions to be determined by solving the following system:

$$\begin{cases} c_1'(x)y_1 + c_2'(x)y_2 = 0 \\ c_1'(x)y_1' + c_2'(x)y_2' = f(x). \end{cases}$$

Example 3.4.6. *We want to solve the following non-homogeneous equation:*

$$y'' + 6y' + 5 = e^{2x}. \quad (1)$$

For this, we consider the associated homogeneous equation:

$$y'' + 6y' + 5 = 0. \quad (2)$$

The general solution of equation (2) is given by:

$$y = C_1e^{-5x} + C_2e^{-x}, \quad C_1, C_2 \in I\!R.$$

To find the general solution of ODE (1) in the form $y = C_1(x)e^{-5x} + C_2(x)e^{-x}$, we solve the following system:

$$\begin{cases} C_1'(x)e^{-5x} + C_2'(x)e^{-x} = 0 \\ -5C_1'(x)e^{-5x} - C_2'(x)e^{-x} = e^{2x} \end{cases}$$

Thus, we find $C_1(x) = -\frac{1}{28}e^{7x} + k_1$ and $C_2(x) = -\frac{1}{12}e^{3x} + k_2$ with $k_1, k_2 \in I\!R$. Therefore, the general solution of ODE (1) is:

$$y = \frac{1}{28}e^{2x} + k_1e^{-5x} + k_2e^{-x}.$$

Special Cases:

Let $\lambda \in I\!R$, and $P_n(x)$ be a polynomial of degree n. Consider the ODE:

$$y'' + ay' + by = P_n(x)e^{\lambda x} \quad (*)$$

The particular solutions are given by:

1) *If λ is not a root of the characteristic equation $r^2 + ar + b = 0$, then*

$$y_p = Q_n(x)e^{\lambda x}, \quad \deg(P_n) = \deg(Q_n)$$

is a particular solution of ODE (*).

2) If λ is a simple root of the characteristic equation $r^2 + ar + b = 0$, then

$$y_p = xQ_n(x)e^{\lambda x}, \quad \deg(P_n) = \deg(Q_n)$$

is a particular solution of ODE (*).

3) If λ is a double root of the polynomial $r^2 + ar + b = 0$, then

$$y_p = x^2Q_n(x)e^{\lambda x} \quad deg(P_n) = deg(Q_n)$$

is a particular solution of the ODE $(*)$.

Example 3.4.7. *1) Consider the equation $y'' + y' - y = x$. $\lambda = 0$ is a simple root of the polynomial $r^2 + r - 1 = 0$, so $y_p = ax^2 + bx + c$ is a particular solution of $y'' + y' - y = x$. $y_p' = 2ax + b$ and $y_p'' = 2a$. By substitution into the ODE, we have:*

$$-ax^2 + (2a - b)x + 2a + b - c = x$$

By identification, we find:

$$a = 0, \quad b = -1, \quad and \quad c = \frac{3}{2}.$$

Thus, $y_p = \frac{1}{6}x^3 - \frac{5}{4}x^2 + \frac{3}{2}x$.

2) Consider the equation $y'' + 2y' = x^2 - 4x + 3$. $\lambda = 0$ is a simple root of the polynomial $r^2 + 2r = 0$, so $y_p = x(ax^2 + bx + c)$ is a particular solution of $y'' + 2y' = x^2 - 4x + 3$. $y_p' = 3ax^2 + 2bx + c$ and $y_p'' = 6ax + 2b$. If we substitute this into the ODE, we obtain:

$$6ax^2 + (6a + 4b)x + 2c = x^2 - 4x + 3$$

By identification, we find:

$$a = \frac{1}{6}, \quad b = -\frac{5}{4}, \quad and \quad c = \frac{3}{2}.$$

Thus, $y_p = \frac{1}{6}x^3 - \frac{5}{4}x^2 + \frac{3}{2}x$.

3) Consider the equation $y'' - 2y' + y = (x^2 + 1)e^x$. $\lambda = 1$ is a double root of the polynomial $r^2 - 2r + 1 = 0$, so $y_p = x^2(ax^2 + bx + c)e^x$ is a particular solution of $y'' - 2y' + y = (x^2 + 1)e^x$.

$y_p' = (4ax^3 + 3bx^2 + 2cx)e^x + y_p$ and $y_p'' = (12ax^2 + 6bx + 2c)e^x - y_p$. If we substitute this into the ODE, we obtain:

$$6ax^2 + (6a + 4b)x + 2c = x^2 - 4x + 3$$

By identification, we find:

$$a = \frac{1}{6}, \quad b = -\frac{5}{4}, \quad \textit{and} \quad c = \frac{3}{2}.$$

Thus, $y_p = \frac{1}{6}x^3 - \frac{5}{4}x^2 + \frac{3}{2}x$.

3.5 Exercises:

Exercise 01:

Consider the function f defined on $\mathbb{R}^*$ and $C, D \in \mathbb{R}$:

$$\tilde{f}(x) = \begin{cases} Ce^{\frac{-1}{x}} & if \quad x > 0 \\ De^{\frac{-1}{x}} & if \quad x < 0 \end{cases}$$

1) *Give a necessary and sufficient condition on C and D for $\tilde{f}(x)$ to be continuous at 0.*

2) *Show that if this condition is met, then this extension is also differentiable at 0 and its derivative is continuous at 0.*

3) *Consider the differential equation $x^2y' - y = 0...(*)$.*
   a) Solve the ODE $(*)$ on the interval on $\mathbb{R}^*$.
   b) Deduce the solution of the ODE $(*)$ on $\mathbb{R}$.

Exercise 02

Give a differential equation whose solutions are of the form:

$$x \to \frac{c+x}{1+x^2}, \quad c \in \mathbb{R}.$$

Exercise 03

1) Find differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy:

$$\forall x \in \mathbb{R}, \quad f'(x) + f(x) = \int_0^1 f(t)\,dt.$$

2) Find differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy:

$$\forall x \in \mathbb{R}, \quad f'(x) + f(x) = f(0) + f(1).$$

Exercise 04

Determine differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy for all $s, t \in \mathbb{R}$:

$$f(s+t) = f(s)f(t).$$

Exercise 05

Solve the ODE:

$$y' + \frac{1}{x}y = \frac{x}{x-1}$$

with $y(2) = 2$.

Exercise 06

Solve the ODE:

$$y'' + 2y' + 4y = xe^x$$

with initial conditions $y(0) = 1$ *and* $y(1) = 0$.

Exercise 07

Using the variable change $z = y'$, *solve on* $]-1, +\infty[$ *the ODE:*

$$(1+x)^2 y'' + (1+x)y' - 2 = 0.$$

3.6 Solution:

Solution of Exercise 1

1) $\lim_{x \to 0^+} f(x) = 0$ *independently of* C *and* $\lim_{x \to 0^-} f(x) = 0$ *only when* $D = 0$*. Thus, we have a continuous extension at 0 if and only if* $D = 0$*, defined by:*

$$\tilde{f}(x) = \begin{cases} Ce^{-\frac{1}{x}} & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

2)

$$\frac{\tilde{f}(x) - \tilde{f}(0)}{x} = \frac{C}{x} e^{-\frac{1}{x}}.$$

Let $u = \frac{1}{x}$*, then* $\lim_{u \to +\infty} ue^{-u} = 0$*.*
Therefore, $\tilde{f}$ *is differentiable and* $\tilde{f}'(0) = 0$ *and*

$$\tilde{f}'(x) = \begin{cases} \frac{C}{x^2} e^{-\frac{1}{x}} & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

Using the same variable change, we get $\lim_{x \to 0} \tilde{f}'(x) = 0$*, so* f' *is continuous at* 0*.*

3)

a) $x^2 y' - y = 0 <=> x^2 y' = y <=> \frac{dy}{y} = \frac{1}{x^2}$*. By integrating, we get:*

$$y(x) = Ce^{-\frac{1}{x}}, \forall x \neq 0 \text{ with } C \in R.$$

b) Let z *be a solution on* R*, its restriction to* $I\!R^*$ *is also a solution and it is equal to* $y = Ce^{-\frac{1}{x}}$*. So, according to the above, for* y *and* y' *to be continuous at* 0*,* C *must be zero if* $x < 0$*, so the solution of the ODE* ($\star$) *is equal to* $\tilde{f}$*.*

Solution of Exercise 2

$$\forall x \in \mathbb{R}, \quad f(x) = \frac{c + x}{1 + x^2} <=> c = (1 + x^2) f(x) - x.$$

By derivation, we have:

$$(1 + x^2) f'(x) + 2x f(x) - 1 = 0.$$

This is the desired ordinary differential equation.

Conversely, consider the ODE $(1 + x^2)y' + 2xy - 1 = 0 ... (\star)$*.*
We verify that the functions $f(x) = \frac{c+x}{1+x^2}$ *with* $c \in \mathbb{R}$ *are its solutions.*

By derivation, we have

$$f'(x) = \frac{-x^2 - 2cx + 1}{(1+x^2)^2}.$$

And if we replace in the ODE ($\star$), we easily verify that f is a solution.

Solution of Exercise 3 *1) Applying the same reasoning, we have:*

$$\int_0^1 f(t)\,dt = c.$$

So,

$$c + d(1 - e^{-1}) = c.$$

Therefore, $d = 0$, and the only solutions are constant functions.

Conversely, it is easy to verify that constant functions are solutions to the equation $f'(x) + f(x) = \int_0^1 f(t)\,dt$.

2) Let f be a solution to the linear homogeneous equation $y' + y = c$ with $c \in \mathbb{R}$. We know that the solutions to this ODE are of the form $f(x) = c + de^{-x}$ with $c, d \in \mathbb{R}$. So, for f to satisfy the ODE $f'(x) + f(x) = f(0) + f(1)$, we must have $f(0) + f(1) = c$, hence:

$$2c + d(1 + e^{-1}) = c$$

$$c = -d(1 + e^{-1}).$$

Thus, the solutions to the equation $f'(x) + f(x) = f(0) + f(1)$ are of the form:

$$f(x) = -d(1 + e^{-1}) + de^{-x}.$$

Conversely, we have $f'(x) = -de^{-x}$, and we verify that f is a solution to the original equation.

Solution of Exercise 4 *For $s = t = 0$, we have $f^2(0) = f(0)$, so $f(0) = 0$ or $f(0) = 1$.*

If $f(0) = 0$, for $s = 0$ and $t \in \mathbb{R}$, we have:

$$f(t) = f(0)f(t) = 0.$$

We can then assume $f(0) = 1$.

Now fix $s \in \mathbb{R}$ and differentiate with respect to $t \in \mathbb{R}$. We have:

$$f'(s+t) = f(s)f'(t).$$

For $t = 0$, we get:

$$f'(s) = f(s)f'(0).$$

Thus, f is a solution to the homogeneous linear differential equation $y' = ay$ with $a \in \mathbb{R}$, whose solution is of the form $f(x) = ce^{ax}$, and since $f(0) = 1$, then $c = 1$.

Solution of Exercise 5 *The ODE $y' + \frac{1}{x}y = \frac{x}{x-1}$ is a non-homogeneous linear first-order ODE.*

We solve the homogeneous case:

$$y' + \frac{1}{x}y = 0.$$

The solution is:

$$y = \frac{c}{x}, \quad c \in \mathbb{R}.$$

We then look for the general solution of the non-homogeneous ODE in the form:

$$y = \frac{c(x)}{x},$$

where c is a function to be determined.

We have:

$$y' = \frac{c'(x)x - c(x)}{x^2}.$$

Substituting into the non-homogeneous ODE, we find:

$$c'(x) = \frac{x^2}{x-1},$$

so

$$c(x) = \frac{x^2}{2} + x + \ln(x-1) + k, \quad k \in \mathbb{R}.$$

Thus, $y = \frac{x}{2} + 1 + \frac{\ln(x-1)}{x} + \frac{k}{x}$.

The condition $y(2) = 2$ gives $k = 0$, hence the solution to the non-homogeneous ODE $y' + \frac{1}{x}y = \frac{x}{x-1}$ is $y = \frac{x}{2} + 1 + \frac{\ln(x-1)}{x}$.

Solution of Exercise 6 *First, solve the homogeneous ODE:*

$$y'' + 2y' + 4y = 0.$$

The characteristic equation is:

$$r^2 + 2r + 4 = 0,$$

with discriminant $\Delta = -12$ *and roots* $-1 \pm i\sqrt{3}$.

The solutions to the homogeneous equation are:

$$y = ae^{-x}\cos(\sqrt{3}x) + be^{-x}\sin(\sqrt{3}x), \quad a, b \in \mathbb{R}.$$

Now, find a particular solution of the non-homogeneous ODE of the form $y_p(x) = (cx+d)e^x$. *We have:*

$$y_p''(x) + 2y_p'(x) + 4y_p(x) = (7cx + 7d + 4c)e^x.$$

By identification, we find $c = \frac{1}{7}$ *and* $d = -\frac{4}{49}$.

Thus, the solutions to the non-homogeneous ODE are functions of the form:

$$y = ae^{-x}\cos(\sqrt{3}x) + be^{-x}\sin(\sqrt{3}x) + \frac{xe^x}{7} - \frac{4e^x}{49}, \quad a, b \in \mathbb{R}.$$

- *If* $y(0) = 1$, *then* $a - \frac{4}{49} = 1$, *so* $a = \frac{53}{49}$.
- *If* $y(1) = 0$, *then* $b = -\frac{53\cos(\sqrt{3}) + 3\sin(\sqrt{3})}{49\sin(\sqrt{3})}$.

Solution of Exercise 7: *If* $z = y'$, *we obtain the ODE:*

$$(1+x)^2 z' + (1+x)z - 2 = 0.$$

which is a non-homogeneous linear first-order ODE. First, solve the homogeneous case:

$$(1+x)^2 z' + (1+x)z = 0,$$

the solution is $z = \frac{c}{1+x}$ *with* $c \in \mathbb{R}$. *Now, find the general solution to the non-homogeneous case using the method of variation of parameters:*

$$z = \frac{c(x)}{1+x},$$

where c *is a function to be determined. We have:*

$$z' = \frac{c'(x)(1+x) - c(x)}{(1+x)^2}.$$

Substituting into the homogeneous ODE, we find:

$$c'(x)(1+x) = 2,$$

so

$$c(x) = 2\ln(1+x) + d, \quad d \in \mathbb{R},$$

and

$$z = \frac{2\ln(1+x)}{1+x} + \frac{k}{1+x}, \quad d \in \mathbb{R}.$$

Thus, the general solution to the initial ODE is:

$$y(x) = (\ln(1+x))^2 + d\ln(1+x) + e, \quad d, e \in \mathbb{R}.$$

Chapter 4

Functions of Two Variables

Definition 4.0.1. *Let U and V be two domains in $I\!R$. A real function of two variables is an object that associates with every pair of real numbers (x, y) at most one real number. Such a function is denoted as:*

$$U \times V \to I\!R$$

$$(x, y) \mapsto f(x, y)$$

Example 4.0.2.

$$f(x, y) = xy^2, \quad f(x, y) = \ln(x + y), \quad f(x, y) = \sqrt{x}e^y$$

4.1 Domain of Definition:

Definition 4.1.1. *The domain of definition of a function f of two variables is the set defined by:*

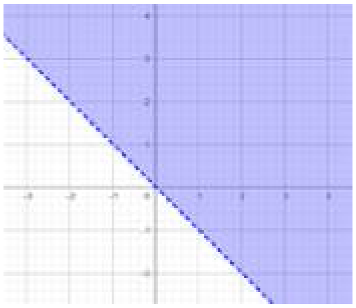
$$D_f = \{(x, y) \in I\!R^2 : f(x, y) \in I\!R\}$$

The determination of the domain of definition of a two-variable function is often done graphically.

Example 4.1.2.

$$f_1(x,y) = xy^2, \quad D_1 = \mathbb{R}^2$$

$$f_2(x,y) = \ln(x+y), \quad D_2 = \{(x,y) \in \mathbb{R}^2 : x > -y\}$$

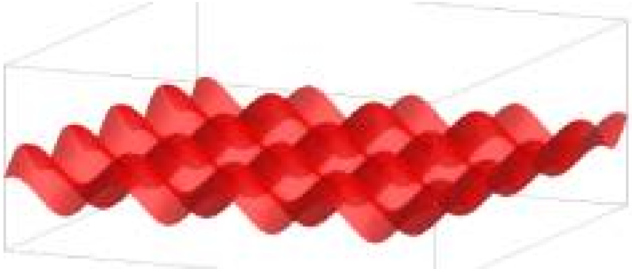


4.2 Graph of a function of two variables

Let $f : D_f \subset \mathbb{R}^2 \to \mathbb{R}$ *be a function of two variables. The graph* G_f *of* f *is the subset of* $\mathbb{R}^3$ *formed by points with coordinates* $(x, y, f(x, y))$*, where* (x, y) *belongs to the domain* D_f*. Therefore, the graph is defined as:*

$$G_f = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D_f \text{ and } z = f(x,y)\}.$$

Example 4.2.1. $f(x,y) = cos(x) + sin(y)$



4.3 Limit Calculation

The limit calculation of a function of two variables can be approached in several ways. To calculate limits, we use general theorems such as operations on limits and bounding. These are the same statements as for single-variable functions; there are no difficulties or new elements.

Proposition 4.3.1. *Let* $f, g : \mathbb{R}^2 \to \mathbb{R}$ *be defined in a neighborhood of* $x_0 \in \mathbb{R}^n$ *such that* f *and* g *have limits at* x_0*. Then:*

$$\lim_{x \to x_0} (f + g) = \lim_{x \to x_0} f + \lim_{x \to x_0} g$$

$$\lim_{x \to x_0} (f \cdot g) = \lim_{x \to x_0} f \cdot \lim_{x \to x_0} g$$

And if g *does not vanish in a neighborhood of* x_0*:*

$$\lim_{x \to x_0} \frac{1}{g} = \frac{1}{\lim_{x \to x_0} g}$$

$$\lim_{x \to x_0} \frac{f}{g} = \frac{\lim_{x \to x_0} f}{\lim_{x \to x_0} g}$$

Example 4.3.2.

$$\lim_{(x,y) \to (0,1)} xy^2 = 0, \qquad \lim_{(x,y) \to (0,\pi)} (\sin(x) + \cos(y)) = -1, \qquad \lim_{(x,y) \to (0,0^+)} x + \ln(y) = -\infty$$

Remark 4.3.3.. —

- *The results above also hold for infinite limits with the usual conventions:*

$$l + \infty = +\infty, \quad l - \infty = -\infty,$$

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty, \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty,$$

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0,$$

$$\ell \times \infty = \infty \quad (l \neq 0), \quad \infty \times \infty = \infty \quad \textit{(with sign multiplication rule).}$$

- *The indeterminate forms are:* $+\infty - \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, ∞^0, 1^∞, *and* 0^0.

Composition is also often useful:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ *be a function of several variables such that* $\lim_{x\to x_0} f(x) = \ell$, *let* $g : \mathbb{R} \to \mathbb{R}$ *be a function of one variable such that* $\lim_{t\to \ell} g(t) = \ell'$, *then the function* $g \circ f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $(g \circ f)(x) = g(f(x))$ *satisfies*

$$\lim_{x\to x_0} (g \circ f)(x) = l'.$$

Theorem 4.3.4.. — ***(Theorem of Sandwich (or Squeeze Theorem))***

Let $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ *be three functions defined in a neighborhood* U *of* $x_0 \in \mathbb{R}^2$. *If for all* $x \in U$, $f(x) \leqslant h(x) \leqslant g(x)$, *then* $\lim_{x\to x_0} f(x) = L$ *and* $\lim_{x\to x_0} g(x) = L$ *imply* $\lim_{x\to x_0} h(x) = L$.

Example 4.3.5. *We want to calculate* $\lim_{(x,y)\to(0,0)} x^2 \sin\left(\frac{1}{xy}\right)$.
It is known that $\forall (x, y) \in \mathbb{R}^2, \quad -1 \leqslant \sin\left(\frac{1}{xy}\right) \leqslant 1$, *hence*

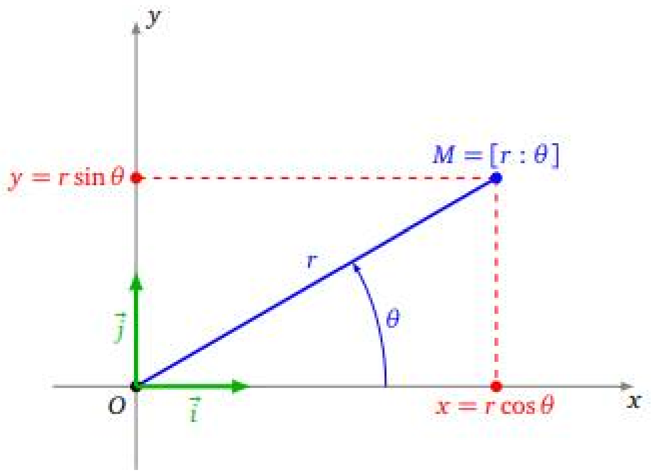
$$-x^2 \leqslant x^2 \sin\left(\frac{1}{xy}\right) \leqslant x^2, \quad \forall (x, y) \in \mathbb{R}^2.$$

Since $\lim_{(x,y)\to(0,0)} x^2 = 0$, *it follows that* $\lim_{(x,y)\to(0,0)} x^2 \sin\left(\frac{1}{xy}\right) = 0$.

4.3.1 Polar Coordinates

Let M *be a point in the plane* $\mathbb{R}^2$. *Let* $O = (0, 0)$ *be the origin. Consider* $(O, \vec{i}, \vec{j})$ *as a right-handed orthonormal coordinate system.*

- *Let* $r = \sqrt{x^2 + y^2}$ *denote the distance from* M *to the origin.*
- *Let* θ *denote the angle between* $\vec{i}$ *and* $\overrightarrow{OM}$.



We denote by $(r:\theta)$ the polar coordinates of the point M. In this course, r will always be positive. The angle θ is not uniquely determined; multiple choices are possible. To ensure uniqueness, we can restrict θ to the interval $[0,2\pi)$ or $(-\pi,\pi]$. Polar coordinates are generally not assigned to the origin (the angle would not be meaningful). In cases of indeterminate form, we can calculate $\lim_{(x,y)\to(0,0)} f(x,y)$ using the change of variables $x=r\cos(\theta)$ and $y=r\sin(\theta)$ with $r>0$ and $\theta\in[0,2\pi]$ (or $\theta\in[-\pi,\pi]$), giving us:

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{r\to 0} f(r\cos(\theta),r\sin(\theta))$$

Polar Coordinates to Cartesian Coordinates:

Cartesian coordinates (x,y) can be obtained from polar coordinates (r,θ) using the formulas

$$x=r\cos\theta \quad \textit{and} \quad y=r\sin\theta.$$

In other words, we define a mapping:

$$(0,+\infty)\times[0,2\pi)\to\mathbb{R}^2, \quad (r,\theta)\mapsto(r\cos\theta,r\sin\theta).$$

Cartesian Coordinates to Polar Coordinates:

Polar coordinates (r,θ) can be obtained from Cartesian coordinates (x,y) using the following formulas:

$$r=\sqrt{x^2+y^2}$$

and, in the case $x>0$ and $y\geqslant 0$,

$$\theta=\arctan\left(\frac{y}{x}\right).$$

When considering functions $f:E\subset\mathbb{R}^2\to\mathbb{R}$, it is sometimes easier to prove results about limits, continuity, etc., by using polar coordinates.

Proposition 4.3.6. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function defined in a neighborhood of $(0,0)\in\mathbb{R}^2$, possibly except at $(0,0)$. If*

$$\lim_{r\to 0} f(r\cos\theta,r\sin\theta)=\ell\in\mathbb{R}$$

exists independently of θ, then

$$\lim_{(x,y)\to(0,0)} f(x,y)=\ell$$

Remark 4.3.7.. —

To clarify this proposition and explain the different practical cases of the limit, here is how it can be approached. We express $f(x, y)$ *in polar coordinates by calculating* $f(r\cos\theta, r\sin\theta)$.

1. *If* $\lim_{r\to 0} f(r\cos\theta, r\sin\theta)$ *exists and is independent of* θ, *then this limit is the limit of* f *at the point* $(0, 0)$.
2. *If* $\lim_{r\to 0} f(r\cos\theta, r\sin\theta)$ *does not exist (or the limit is not finite), then* f *does not have a finite limit at the point* $(0, 0)$.
3. *If* $\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = \ell(\theta)$ *depends on* θ, *then* f *does not have a limit at the point* $(0, 0)$. *To justify this, we provide two values* θ_1 *and* θ_2 *such that* $\ell(\theta_1) \neq \ell(\theta_2)$.

Example 4.3.8.

$$1)\ \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3(\theta)}{r^2(\cos^2(\theta)+\sin^2(\theta))} = \lim_{r\to 0} \frac{r\cos^3(\theta)}{1} = 0 \quad \textit{for } \theta \in [0, 2\pi]$$

2)

$$\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^3} = \lim_{r\to 0} \frac{r\sin\theta}{r^2(\cos^2\theta + r\sin^3\theta)} = \frac{1}{r}\cdot\frac{\sin\theta}{\cos^2\theta + r\sin^3\theta}.$$

Let's fix θ *such that* $\sin\theta \neq 0$ *(i.e.,* $\theta \neq 0, \pi, 2\pi$*)). Then, as* $r \to 0$, $f(r\cos\theta, r\sin\theta)$ *does not have a finite limit. In particular, the function* $(x, y) \mapsto f(x, y)$ *does not have a finite limit at* $(0, 0)$.

3)

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^2\cos(2\theta)}{r^2} = \cos(2\theta)$$

which depends on θ, *so the limit does not exist.*

Remark 4.3.9.. —

If $f : \mathbb{R}^2 \to \mathbb{R}$ *is a function such that for each fixed* θ, $\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = \ell$, *then we cannot conclude that* f *has* ℓ *as its limit at the point* $(0, 0)$.

Example 4.3.10. *Consider the function f defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ by*

$$f(x,y) = \frac{xy^2}{x^2+y^4}.$$

Along every ray, f tends to 0, meaning for each fixed θ,

$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = 0,$$

however, f does not have a limit at $(0,0)$.
Indeed, $f(r\cos\theta, r\sin\theta) = \frac{r\cos\theta\sin^2\theta}{\cos^2\theta + r^2\sin^4\theta}$.

Let's fix θ and discuss according to its value:

- *If $\cos\theta \neq 0$, then the numerator tends to 0 while the denominator tends to $\cos^2\theta \neq 0$. Therefore,*
$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = 0.$$
- *If $\cos\theta = 0$, then (x,y) lies on points where $x = 0$, and hence $f(r\cos\theta, r\sin\theta) = f(0,y) = 0$.*

In all cases, f tends to 0 along all rays defined by a fixed angle θ.
In the next section, we will provide a method to demonstrate the non-existence of this limit.

Remark 4.3.11.. —

If the pair (x,y) tends to $(x_0,y_0) \in\ R^2-(0,0)$, we can set $X = x - x_0$ and $Y = y - y_0$. If the pair (x,y) tends to (∞,∞), we can set $X = \dfrac{1}{x}$ and $Y = \dfrac{1}{y}$.

4.3.2 Method of Paths

Proposition 4.3.12. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined in a neighborhood of $(x,y) \in \mathbb{R}^2$, possibly except at (x,y).*

1. *If f has a limit l at the point (x,y), then the restriction of f to any curve passing through (x_0,y_0) has a limit at (x_0,y_0), and this limit is l.*

2. Conversely, if the restrictions of f to two curves passing through (x_0, y_0) have different limits at (x_0, y_0), then f does not have a limit at (x_0, y_0).

Remark 4.3.13.. —

The previous proposition shows that $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist, we can consider two different paths $y_1 = \phi_1(x) \to y_0$ and $y_2 = \phi_2(x) \to y_0$. If

$$\lim_{x\to x_0} f(x, \phi_1(x)) \neq \lim_{x\to x_0} f(x, \phi_2(x)),$$

then the limit does not exist.

Example 4.3.14. *Consider the limit:*

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}.$$

Let $y_1 = x$ and $y_2 = \sqrt{x}$. Both y_1 and y_2 approach 0 as x approaches 0.

$$f(x, y_1) = f(x, x) = \frac{x}{1+x} \to 0 \quad \textit{as} \quad x \to 0.$$

$$f(x, y_2) = f(x, \sqrt{x}) = \frac{x^2}{2x^2} = \!\!\to \frac{1}{2} \quad \textit{as} \quad x \to 0.$$

Since these limits are different, $\lim_{(x,y)\to(0,0)} \frac{x-y^2}{x^2+y^4}$ does not exist.

4.3.3 Method of Sequences

Proposition 4.3.15. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined in a neighborhood of $(x, y) \in \mathbb{R}^2$, possibly except at (x, y). If f has a limit ℓ at (x_0, y_0), then for any sequence $(x_n, y_n) \in D_f$ such that $(x_n, y_n) \to (x_0, x_0)$, we have $f(x_n, y_n) \to \ell$ when $n \to +\infty$.*

Remark 4.3.16.. —

The previous proposition show that $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist, we can consider two sequences $(x_n, y_n)- > (0,0)$ and (x'_n, y'_n) both approaching (x_0, y_0).

If

$$\lim_{n\to+\infty} f(x_n, y_n) \neq \lim_{n\to+\infty} f(x'_n, y'_n),$$

then the limit does not exist.

Example 4.3.17. *Consider the limit:*

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}.$$

Let $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$ *and* $(x'_n, y'_n) = \left(\frac{1}{n}, \frac{2}{n}\right)$.

We have:

$$f(x_n, y_n) \to 0 \quad \textit{and} \quad f(x'_n, y'_n) \to -\frac{3}{5} \ \textit{when } n \to +\infty.$$

Since these limits are different, the limit $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ *does not exist.*

4.3.4 Iterated Limits

If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = l \in I\!R$, *then* $\lim\limits_{x\to x_0} \lim\limits_{y\to y_0} f(x,y) \neq \lim\limits_{y\to y_0} \lim\limits_{x\to x_0} f(x,y)$.

Conversely, if $\lim\limits_{x\to x_0} \lim\limits_{y\to y_0} f(x,y) \neq \lim\limits_{y\to y_0} \lim\limits_{x\to x_0} f(x,y)$, *then* $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ *does not exist.*

Example 4.3.18. *We want to show that the function* f *defined by:*

$$f(x,y) = \frac{x^3-y^3}{x^3+y^3}$$

does not have a limit at $(0,0)$.

We have:

$$\lim_{y\to 0} \lim_{x\to 0} f(x,y) = -1$$

and

$$\lim_{x\to 0} \lim_{y\to 0} f(x,y) = 1.$$

Since these two limits are different, the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ *does not exist.*

Remark 4.3.19.. —

If the two iterated limits are equal, we cannot draw any conclusion.

4.4 Continuity

Definition 4.4.1. *Let f be a function of two variables and (x_0, y_0) a point in D_f. f is said to be continuous at (x_0, y_0) if:*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

Proposition 4.4.2. *Let f and g are two functions continuous with two variables continious in $(x_0, y_0) \in D_f \cap D_g$, then:*

- *$f + g$ is continuous at (x_0, y_0),*
- *$f \cdot g$ and $\frac{f}{g}$ (where $g(x, y) \neq 0$ in a neighborhood of (x_0, y_0)) are continuous at (x_0, y_0),*
- *If $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $h \circ f$ is continuous at (x_0, y_0).*

Remark 4.4.3.. —

The mappings defined by $(x, y) \mapsto x + y$, $(x, y) \mapsto x^2 + xy$, as well as all polynomial functions in two variables x and y, are continuous on $\mathbb{R}^2$ (for example, $(x, y) \mapsto x^2+3xy$). Similarly, all rational functions in two variables are continuous where they are defined. Since the exponential function is continuous, $(x, y) \mapsto e^{x-y}$ is continuous on $\mathbb{R}^2$. The function defined by $f(x, y) = \frac{x-y}{x^2+y^2}$ is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Example 4.4.4. *Consider the function f defined on $I\!R^2$ by:*

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

f is continuous on $R^2\{(0, 0)$ as a quotient of continuous functions. Continuity at $(0, 0)$:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0} r\cos^3(\theta) = 0 = f(0,0).$$

Therefore, f is continuous at $(0, 0)$.

4.5 Continuity Extension

Let f be a function of two variables and (x_0, y_0) a point not in D_f. f is said to be continuously extendable at (x_0, y_0) if $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = l \in I\!R$, and we can define the function $\tilde{f}$ as follows:

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \neq (x_0, x_0), \\ l & \text{if } (x, y) = (x_0, y_0). \end{cases}$$

Example 4.5.1. *1) Consider the function f defined on $I\!R^2 - \{(0, 0)\}$ by $f(x, y) = \frac{xy^2}{x^2+y^2}$. We have $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{r\to 0} \frac{r\cos(\theta)\sin^2(\theta)}{r} = 0$. Therefore, f has a continuity extension at $(0, 0)$ and we define:*

$$\tilde{f}(x, y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

2) Consider the function f defined on $I\!R^2 - \{(0, 0)\}$ by $f(x, y) = \frac{xy}{x^2+y^2}$. We have $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{r\to 0} \frac{r\cos(\theta)\sin(\theta)}{r} = \cos(\theta)\sin(\theta)$, which does not exist. Thus, f does not have a continuity extension at $(0, 0)$.

4.6 Differentiability

Definition 4.6.1. *Let f be a function of two variables and $(x_0, y_0) \in D_f$. f is said to be differentiable at (x_0, y_0) if there exists a continuous linear map df such that:*

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(x_0 + h_1, y_0 + h_2) - f(x_0, y_0) - df(x_0, y_0)(h_1, h_2)}{\sqrt{h_1^2 + h_2^2}} = 0.$$

Remark 4.6.2.. —

1) The function df is called the differential of f at (x_0, y_0).

2) In practice, to show that f is differentiable at (x_0, y_0), we demonstrate:

$$f(x_0 + h_1, y_0 + h_2) - f(x_0, y_0) = df(x_0, y_0)(h_1, h_2) + o(\sqrt{h_1^2 + h_2^2})$$

where

$$\lim_{(h_1,h_2)\to(0,0)} \frac{o(\sqrt{h_1^2 + h_2^2})}{\sqrt{h_1^2 + h_2^2}} = 0$$

Example 4.6.3. *Consider the function f defined on $\mathbb{R}^2$ by $f(x, y) = xy$. We want to show that f is differentiable on $\mathbb{R}^2$. For any $(x_0, y_0) \in \mathbb{R}^2$, we have:*

$$f(x_0 + h_1, y_0 + h_2) - f(x_0, y_0) = y_0 h_1 + x_0 h_2 + h_1 h_2.$$

$$\lim_{(h_1,h_2)\to(0,0)} \frac{h_1 h_2}{\sqrt{h_1^2 + h_2^2}} = \lim_{r\to 0} r\cos(\theta)\sin(\theta) = 0.$$

Therefore, f is differentiable on $\mathbb{R}^2$ and $df(x_0, y_0)(h_1, h_2) = y_0 h_1 + x_0 h_2$ for all $(x, y) \in \mathbb{R}^2$.

Proposition 4.6.4. *Let f be a function of two variables and $(x_0, y_0) \in D_f$.*
If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).
Conversely,
If f is not continuous at (x_0, y_0), then f is not differentiable at (x_0, y_0).

Example 4.6.5. *Consider the function f defined by*

$$\tilde{f}(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0} \frac{r\sin(\theta) r\cos(\theta)}{r^2\cos^2(\theta) + r^2\sin^2(\theta)} = \frac{\sin(\theta)\cos(\theta)}{\cos^2(\theta) + \sin^2(\theta)}.$$

The limit does not exist, so f is not continuous at $(0, 0)$, hence f is not differentiable at $(0, 0)$.

Remark 4.6.6.. —

The sum, product, quotient, scalar multiple, and composition of differentiable functions are differentiable.

Definition 4.6.7. *The partial derivatives of f are the functions:*

$$(x, y) \mapsto \frac{\partial f(x, y)}{\partial x}, \quad \text{and} \quad (x, y) \mapsto \frac{\partial f(x, y)}{\partial y}.$$

In practice, to compute a partial derivative, we differentiate with respect to one variable while treating the other as constant.

Remark 4.6.8.. —

For piecewise-defined functions, the partial derivatives at a point $(x_0, y_0) \in D_f$*, if they exist, are given by the following limit formulas:*

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}, \quad \frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.$$

Remark 4.6.9.. —

We can also use the notations

$$(x, y) \to \frac{df(x, y)}{dx}, \quad \text{and} \quad (x, y) \to \frac{df(x, y)}{dy}$$

or

$$(x, y) \to f'_x(x, y), \quad \text{and} \quad (x, y) \to f'_y(x, y)$$

Example 4.6.10. *Let's compute the partial derivatives of the function* f *defined by* $f(x, y) = x^2y$*. We have:*

$$\frac{\partial f(x, y)}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f(x, y)}{\partial y} = x^2.$$

Proposition 4.6.11. *Let* f *be a function of two variables and* (x_0, y_0) *a point in* D_f*. If* f *is differentiable at* (x_0, y_0)*, then* $\frac{\partial f(x,y)}{\partial x}$ *and* $\frac{\partial f(x,y)}{\partial y}$ *exist, and we have:*

$$df(x, y)(h_1, h_2) = \frac{\partial f(x, y)}{\partial x}h_1 + \frac{\partial f(x, y)}{\partial y}h_2.$$

Example 4.6.12. *We've already shown that* f *is differentiable and* $df(x, y)(h_1, h_2) = yh_1 + xh_2$ *for all* $(x, y) \in I\!R^2$*. Hence,* $\frac{\partial f(x,y)}{\partial x} = y$ *and* $\frac{\partial f(x,y)}{\partial y} = x$*.*

Remark 4.6.13.. —

The converse of previous proposition is false.

Counterexample

Let f *be the function defined by:*

$$\tilde{f}(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

As shown in example 4.6.5, f *is not differentiable. Now let's compute the partial derivatives:*

$$\frac{\partial f(0, 0)}{\partial x} = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0,$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0.$$

Thus, the partial derivatives exist but the function is not differentiable.

Remark 4.6.14.. —

Polynomials, logarithmic functions, exponential functions, trigonometric functions, etc., are all differentiable functions.

Proposition 4.6.15. *Let f be a function of two variables. If $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ exist and are continuous, then f is differentiable.*

Example 4.6.16. *Consider the function:*

$$\tilde{f}(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

On $I\!R^2 \setminus \{(0,0)\}$, f is differentiable as it is the ratio of differentiable functions.

$$\frac{\partial f(x,y)}{\partial x} = \frac{2xy^4}{(x^2+y^2)^2}$$

and

$$\frac{\partial f(x,y)}{\partial y} = \frac{2x^4y}{(x^2+y^2)^2}.$$

$$\frac{\partial f(0,0)}{\partial x} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

and

$$\frac{\partial f(0,0)}{\partial y} = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0.$$

Now, let's study the continuity of the partial derivatives:

$$\lim_{(x,y)\to(0,0)} \frac{\partial f(x,y)}{\partial x} = \lim_{r\to 0} \frac{2r\cos^3(\theta)\sin^4(\theta)}{r^2\cos^2(\theta)+r^2\sin^2(\theta)} = 0.$$

$$\lim_{(x,y)\to(0,0)} \frac{\partial f(x,y)}{\partial y} = \lim_{r\to 0} \frac{2r\sin^3(\theta)\cos^4(\theta)}{r^2\cos^2(\theta)+r^2\sin^2(\theta)} = 0.$$

The partial derivatives exist and are continuous, hence f is differentiable.

Definition 4.6.17. *Let f be a function of two variables and let I . We say that f is of class C^1 in I if it is differentiable and its differential is continuous in I.*

Theorem 4.6.18.. — *L*

et f be a function of two variables.
f is of class C^1 at $(x_0, y_0) \in D_f$ if and only if $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ exist and are continuous at (x_0, y_0).

Example 4.6.19. *Consider the function:*

$$\tilde{f}(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

It has been shown in Example 4.6.16 that the partial derivatives exist and are continuous over $\mathbb{R}^2$, hence f is of class C^1 on $\mathbb{R}^2$.

Definition 4.6.20. *Let f be a function of two variables and $(x_0, y_0) \in D_f$. We say that f is twice differentiable at (x_0, y_0) if it is differentiable and its differential is differentiable at (x_0, y_0).*

Notation

The second-order partial derivatives of a function $f(x,y)$ are defined as follows:

1) $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$

2) $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$

3) $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$

4) $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$

Example 4.6.21. *Let f be the function defined on $\mathbb{R}^2$ by $f(x,y) = x^2y^2$. We compute the first and second-order partial derivatives:*

$$\frac{\partial f}{\partial x}(x,y) = 2xy^2, \quad \frac{\partial f}{\partial y}(x,y) = 2yx^2.$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2y^2, \quad \frac{\partial^2 f}{\partial y^2}(x,y) = 2x^2, \quad \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = 4xy.$$

Remark 4.6.22.. —

Polynomials, logarithmic functions, exponential functions, trigonometric functions, etc., are all twice differentiable functions.

Proposition 4.6.23. *Let f be a function of two variables. If f is twice differentiable, then all second-order partial derivatives exist at (x_0, y_0), and the second differential of f at (x_0, y_0) is the bilinear form given by:*

$$d^2f(x_0,y_0)((h_1,h_2),(k_1,k_2)) = \frac{\partial^2 f}{\partial x^2}(x_0,y_0)h_1k_1+\frac{\partial^2 f}{\partial y^2}(x_0,y_0)h_2k_2+\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)(h_1k_2+h_2k_1).$$

Example 4.6.24. *Reconsider the example defined by $f(x,y) = x^2y^2$. Then:*

$$df(x,y)(h_1,h_2)(k_1,k_2) = 2y^2h_1k_1 + 2x^2h_2k_2 + 4xyh_1k_2 + 4xyh_2k_1.$$

Theorem 4.6.25.. — *L*

et f be a function of two variables. If the second-order partial derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x\partial y}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y\partial x}$$

exist and are continuous in a neighborhood of a point, then f is said to be twice differentiable at that point.

Theorem 4.6.26.. — *(*

Schwarz's Theorem) Let f be a function of two variables. Suppose that the mixed partial derivatives

$$\frac{\partial^2 f(x,y)}{\partial x\partial y} \quad \text{and} \quad \frac{\partial^2 f(x,y)}{\partial y\partial x}$$

exist and are continuous at a point $(x_0, y_0) \in D_f$. Then,

$$\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0) = \frac{\partial^2 f}{\partial y\partial x}(x_0,y_0).$$

Remark 4.6.27.. —

This theorem is often used in its contrapositive form.

Definition 4.6.28. *Let f be a function of two variables. We say that f is of class C^2 if it is twice differentiable and its second differential is continuous.*

Theorem 4.6.29.. — *L*

et f be a function of two variables. The function f is of class $\mathcal{C}^2$ if all second-order partial derivatives

$$\frac{\partial^2 f(x,y)}{\partial x^2}, \quad \frac{\partial^2 f(x,y)}{\partial y^2}, \quad \frac{\partial^2 f(x,y)}{\partial x \partial y}, \quad \text{and} \quad \frac{\partial^2 f(x,y)}{\partial y \partial x}$$

exist and are continuous in a neighborhood of each point in the domain.

Example 4.6.30. *Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by:*

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We will show that f is not of class $\mathcal{C}^2$. To do so, we will prove that $\frac{\partial^2 f}{\partial x \partial y}$ is not continuous at $(0,0)$.

Since f is the quotient of smooth functions (with the denominator never vanishing away from the origin), it is differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$, and we can compute its partial derivatives there.

We first compute $\frac{\partial f}{\partial y}$ on $\mathbb{R}^2 \setminus \{(0,0)\}$:

$$\frac{\partial f}{\partial y}(x,y) = \frac{xy^4 + 3x^3y^2}{(x^2+y^2)^2}.$$

At the origin, since $f(x,0) = 0$ for all x, we have:

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0.$$

We now consider the mixed partial derivative $\frac{\partial^2 f}{\partial x \partial y}$ on $\mathbb{R}^2 \setminus \{(0,0)\}$. Differentiating the expression of $\frac{\partial f}{\partial y}(x,y)$ with respect to x, we obtain:

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{y^8 - 3x^2y^2 + 7x^2y^6 + 3x^4y^4}{(x^2+y^2)^4}.$$

At the origin, we define the mixed derivative by the limit:

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(h,0) - \frac{\partial f}{\partial x}(0,0)}{h} = 0,$$

since $\frac{\partial f}{\partial x}(x,0) = 0$ for all x, and in particular at $x = 0$.

However, to determine whether $f \in \mathcal{C}^2$*, we must check whether* $\frac{\partial^2 f}{\partial x \partial y}$ *is continuous at* $(0,0)$*.*

Let us examine the limit of $\frac{\partial^2 f}{\partial x \partial y}(x,y)$ *as* $(x,y) \to (0,0)$*. Switching to polar coordinates* $x = r\cos\theta,\ y = r\sin\theta$*, we get:*

$$\frac{\partial^2 f}{\partial x \partial y}(r,\theta) = \frac{r^8\left(\sin^8\theta - 3\cos^2\theta\sin^2\theta + 7\cos^2\theta\sin^6\theta + 3\cos^4\theta\sin^4\theta\right)}{r^8}.$$

Simplifying, we obtain:

$$\frac{\partial^2 f}{\partial x \partial y}(r,\theta) = \sin^8\theta - 3\cos^2\theta\sin^2\theta + 7\cos^2\theta\sin^6\theta + 3\cos^4\theta\sin^4\theta,$$

*which **depends on* θ*** and not on* r*. Therefore, the limit*

$$\lim_{(x,y)\to(0,0)} \frac{\partial^2 f}{\partial x \partial y}(x,y)$$

***does not exist**, since it depends on the direction of approach.*

Hence, $\frac{\partial^2 f}{\partial x \partial y}$ *is not continuous at* $(0,0)$*, and we conclude that* $f \notin \mathcal{C}^2$*.*

4.6.1 n-times differentiability

Definition 4.6.31. *Let* f *be a function of two variables.* f *is said to be* n*-times differentiable if it is* $(n-1)$*-times differentiable and the* $(n-1)$*-th differential is differentiable.*

Definition 4.6.32. *Let* f *be a function of two variables.* f *is said to be of class* C^n *if it is* n*-times differentiable and the* n*-th differential is continuous.*

4.7 Taylor's Formula

4.7.1 Taylor's Formula with Lagrange Remainder:

Let f *be a function of two variables of class* C^{n+1}*. For* $(a_1,a_2) \in D_f$ *and* $(a_1+h_1, a_2+h_2) \in D_f$*, we have:*

$$f(a_1+h_1,a_2+h_2) = f(a_1,a_2)+\sum_{k=1}^{n}\frac{1}{k!}d^{(f)}(a_1,a_2)(h_1,h_2)^k+\frac{1}{(n+1)!}d^{(n+1)}f(a_1+\theta h_1,a_2+\theta h_2)(h_1,h_2)^{(n+1)}$$

where $\theta \in]0,1[$*.*

Example 4.7.1. *We want to give Taylor's formula of order 1 for the function f at (0,0) defined by $f(x,y) = e^{x+y} + y - 1$.*

f is of class 2,

$$df(0,0)(h_1,h_2) = h_1 + 2h_2$$

and

$$d^2f(\theta h_1, \theta h_2)^2 = e^{\theta(h_1+h_2)}h_1^2 + e^{\theta(h_1+h_2)}h_2 + 2e^{\theta(h_1+h_2)}h_1h_2$$

Thus,

$$f(h_1,h_2) = h_1 + 2h_2 + e^{\theta(h_1+h_2)}h_1^2 + e^{\theta(h_1+h_2)}h_2 + 2e^{\theta(h_1+h_2)}h_1h_2$$

where $\theta \in]0,1[$.

Taylor's Formula with Young's Remainder

Let f be a function of two variables of class C^n. For (x_0,y_0) a point in D_f, there exists a function $\epsilon : D_f \to \mathbb{R}$ such that:

$$f(a_1+h_1, a_2+h_2) = f(a_1,a_2) + \sum_{k=1}^{n} \frac{1}{k!} d^{(f)}(a_1,a_2)(h_1,h_2)^k + ((h_1^2+h_2)^n \epsilon(h_1,h_2))$$

where $\lim_{(h_1,h_2)\to(0,0)} \epsilon(h_1,h_2) = 0$.

Remark 4.7.2.. — T

his formula gives the Taylor expansion of a function f up to order n.

4.7.2 Local Extrema of a Function of Two Variables

Definition 4.7.3. *Let f be a function of two variables defined on a domain D_f.*
We say that f has a local maximum at $x_0 \in U$ if there exists an open set containing x_0 such that for all $x \in D_f \cap U$, $f(x) \leqslant f(x_0)$. Similarly, we say that f has a local minimum at $x_0 \in U$ if there exists an open set containing x_0 such that for all $x \in D_f \cap U$, $f(x) \geqslant f(x_0)$. We say that f has a local extremum if it has a local minimum or local maximum.

Proposition 4.7.4. *Let f be a function of two variables. If f has a local extremum at a point $M \in D_f$, then:*

$$\frac{df(M)}{dx} = \frac{df(M)}{dy} = 0$$

In this case, M is called a critical point.

Example 4.7.5.

$$f(x,y) = 2 - (x-1)^2 - (y-3)^2$$

We find:

$$\frac{df(x,y)}{dx} = -2(x-1) = 0, \quad \frac{df(x,y)}{dy} = -2(y-3) = 0$$

Thus, $x = 1$ and $y = 3$. Therefore, the only critical point is $M = (1,3)$.
$f(M) = 2$ and $f(x,y) \leqslant f(M)$ over $\mathbb{R}^2$, hence on a neighborhood of M, f has a local maximum at M.

Remark 4.7.6.. —

If $\frac{df(M)}{dx} = \frac{df(M)}{dy} = 0$, f does not necessarily have an extremum at M. For example, consider $f(x,y) = x^2y^3$. We have:

$$\frac{df(x,y)}{dx} = 2xy^3 = 0, \quad \frac{df(x,y)}{dy} = 3x^2y^2 = 0$$

The only critical point is $(0,0)$, yet $f(x,y) - f(0,0) = x^2y^3$ does not have a constant sign, implying f does not have an extremum at $(0,0)$.

Proposition 4.7.7. *Let f be a function of two variables of class $\mathcal{C}^2$ in a neighborhood of a critical point $M \in D_f$. Denote:*

$$R = \frac{\partial^2 f}{\partial x^2}(M), \quad T = \frac{\partial^2 f}{\partial y^2}(M), \quad S = \frac{\partial^2 f}{\partial x \partial y}(M).$$

Then:

1) If $RT - S^2 > 0$ and $R > 0$, then f has a local minimum at M.

2) If $RT - S^2 > 0$ and $R < 0$, then f has a local maximum at M.

3) If $RT - S^2 < 0$, then f has a saddle point at M (i.e., no local extremum).

4) If $RT - S^2 = 0$, the test is inconclusive (degenerate case).

Example 4.7.8. *Let $f(x,y) = x^4 + y^4 - 4xy + 8$. To find the critical points, we solve:*

$$\begin{cases} \dfrac{\partial f}{\partial x} = 4x^3 - 4y = 0 \\ \dfrac{\partial f}{\partial y} = 4y^3 - 4x = 0 \end{cases} \Rightarrow \begin{cases} y = x^3 \\ x = y^3 \end{cases} \Rightarrow y = y^9 \Rightarrow y = 0,\ \pm 1, \quad \text{and } x = y^3.$$

Thus, the critical points are $(0,0)$, $(1,1)$, $(-1,-1)$.

At** $(0,0)$**:

$$R = \frac{\partial^2 f}{\partial x^2}(0,0) = 0, \quad T = \frac{\partial^2 f}{\partial y^2}(0,0) = 0, \quad S = \frac{\partial^2 f}{\partial x \partial y}(0,0) = -4,$$

$$\Rightarrow \quad RT - S^2 = -16 < 0.$$

So f has a saddle point at $(0,0)$.

At** $(1,1)$**:

$$R = \frac{\partial^2 f}{\partial x^2}(1,1) = 12, \quad T = \frac{\partial^2 f}{\partial y^2}(1,1) = 12, \quad S = \frac{\partial^2 f}{\partial x \partial y}(1,1) = -4,$$

$$\Rightarrow \quad RT - S^2 = 144 - 16 = 128 > 0, \quad \text{and } R > 0.$$

So f has a local minimum at $(1,1)$.

***At** $(-1,-1)$**:** The second derivatives are the same as at $(1,1)$, so f also has a local minimum at $(-1,-1)$.*

Example of a Degenerate Case:

Let

$$f(x,y) = x^4 + y^3 - 3y - 2.$$

We compute the first-order partial derivatives:

$$\frac{\partial f}{\partial x} = 4x^3, \quad \frac{\partial f}{\partial y} = 3y^2 - 3.$$

Solving $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, we find the critical points:

$$4x^3 = 0 \Rightarrow x = 0, \quad 3y^2 - 3 = 0 \Rightarrow y = \pm 1.$$

Thus, the critical points are $(0,1)$ and $(0,-1)$.

We now compute the second-order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 12x^2, \quad \frac{\partial^2 f}{\partial y^2} = 6y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

Analysis at $(0,1)$*:*

$$R = \frac{\partial^2 f}{\partial x^2}(0,1) = 0, \quad T = \frac{\partial^2 f}{\partial y^2}(0,1) = 6, \quad S = \frac{\partial^2 f}{\partial x \partial y}(0,1) = 0.$$

$$\Rightarrow \quad D = RT - S^2 = 0.$$

Since $D = 0$*, the second derivative test is inconclusive at* $(0,1)$*.*

To investigate further, we consider the Taylor expansion of f *near* $(0,1)$*. Let* $h_1 = x$*,* $h_2 = y - 1$*, so that* $(x,y) = (h_1, 1 + h_2)$*. Then:*

$$f(h_1, 1+h_2) - f(0,1) = h_1^4 + (1+h_2)^3 - 3(1+h_2) - \big(1^3 - 3 \cdot 1\big) = h_1^4 + h_2^3 + 3h_2^2.$$

Thus,

$$f(h_1, 1+h_2) - f(0,1) = h_1^4 + h_2^3 + 3h_2^2.$$

For small h_2*, the dominant term in* $h_2^3 + 3h_2^2$ *is* $3h_2^2 > 0$*, and since* $h_1^4 \geqslant 0$*, we conclude:*

$$f(h_1, 1+h_2) > f(0,1) \quad \text{for all } (h_1,h_2) \neq (0,0) \text{ sufficiently close to } (0,0).$$

Therefore, f *has a local minimum at* $(0,1)$*, even though the second derivative test was inconclusive.*

4.7.3 Global Extrema of a Two-Variable Function:

Let f *be a function of two variables and let* $(x_0, y_0) \in D_f$*. We say that* f *has a global minimum (respectively, maximum) if* $f(x,y) \geqslant f(x_0,y_0)$ *(respectively,* $f(x,y) \leqslant f(x_0,y_0)$*) for all* $(x,y) \in D_f$*.*

Example 4.7.9. *Consider the function defined on* $\mathbb{R}^2$ *by* $f(x,y) = x^2 + y^2 - 2$*. We have* $f(x,y) - f(0,0) = x^2 + y^2 \geqslant 0$*, thus* f *has a global minimum at* $(0,0)$*.*

Remark 4.7.10.. —

1) *If there exists* $(x_0, y_0) \in D_f$ *such that* $f(x,y) \to +\infty$ *near* (x_0, y_0)*, then* f *does not have a global maximum.*

2) If there exists $(x_0, y_0) \in D_f$ such that $f(x,y) \to -\infty$ near (x_0, y_0), then f does not have a global minimum.

Example 4.7.11. *Consider the function defined on $\mathbb{R}^2$ by $f(x,y) = x^2 + y$.*
We have $\lim\limits_{(x,y)\to(+\infty,0)} f(x,y) = +\infty$, so f does not have a global maximum.
Also, $\lim\limits_{(x,y)\to(0,-\infty)} f(x,y) = -\infty$, so f does not have a global minimum.

4.7.4 Exercise:

Consider the function defined on $\mathbb{R}^2$ *by:*

$$f(x,y) = 2x^3 + 6xy - 3y^2 + 2.$$

Study the existence of local and global extrema of f *on* $\mathbb{R}^2$*.*

Solution:

$\dfrac{df(x,y)}{dx} = 6x^2 + 6y$, $\dfrac{df(x,y)}{dy} = 6x - 6y$ *and the critical points are* $(0,0)$ *and* $(-1,-1)$*.*
Also, $\dfrac{d^2f(x,y)}{dx^2} = 12x$, $\dfrac{d^2f(x,y)}{dy^2} = -6$, *and* $\dfrac{d^2f(x,y)}{dxdy} = 6$.

At $(0,0)$, $R = 0$, $T = -6$, *and* $S = 6$, *so* $RT - S^2 < 0$, *which implies that* f *does not have a local extremum at* $(0,0)$*.*
At $(-1,-1)$, $R = -12$, $T = -6$, *and* $S = 6$, *so* $RT - S^2 > 0$ *and since* $R < 0$, f *has a local maximum at* $(-1,-1)$*.*
$\lim\limits_{(x,y)\to(0,+\infty)} f(x,y) = -\infty$, *so* f *does not have a global minimum.*
$\lim\limits_{(x,y)\to(+\infty,1)} f(x,y) = +\infty$, *so* f *does not have a global maximum.*

4.8 Exercises:

Exercise 01:

Give the domain of definition of the following functions and graphically represent them:
) $f_1(x,y) = \dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{y}} + \ln(1-x-y),\ f_2(x,y) = \dfrac{1}{\sqrt{x^2+y^2-1}} + \sqrt{4-x^2-y^2}$

Exercise 02:

Calculate the following limits:

1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^4}$, 2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{x+y}-1}{x}$, 3) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{|x|+|y|}$,

4) $\displaystyle\lim_{(x,y)\to(0,0)} (x+y)\sin\left(\frac{1}{x^2+y^2}\right)$, 5) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x|+|y|}{x^2+y^2}$, 6) $\displaystyle\lim_{(x,y)\to(0,a)} \frac{\sin(xy)}{y}$,

7) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y}{x^2-y^2}$.

Exercise 03:

Study the continuity of the following functions on their domains of definition:

$$1)\quad f_1(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$$

$$2)\quad f_2(x,y) = \begin{cases} y^2\sin\left(\frac{x}{y}\right) & \textit{if } y \neq 0 \\ 1 & \textit{if } y = 0 \end{cases}$$

$$3)\quad f_3(x,y) = \begin{cases} 2x^2+y^2-1 & \textit{if } x^2+y^2 > 1 \\ x^2 & \textit{otherwise} \end{cases}$$

Exercise 04:

Consider the following function:

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & \textit{if } (x,y) \neq (0,0) \\ 0 & \textit{if } (x,y) = (0,0) \end{cases}$$

1) Calculate $f'_x(0,0)$ *and* $f'_y(0,0)$*.*

2) Show using the definition that f is not differentiable.

3) Are f'_x and f'_y continuous at $(0,0)$?

Exercise 05:

Consider the following function:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

1) Study the continuity of f on its domain of definition.

2) Calculate $\frac{df}{dx}(0,0)$ and $\frac{df}{dy}(0,0)$. What can be deduced?

Exercise 06:

Study the continuity extension of the following functions:

$$f(x,y) = \frac{\sin(x) - \sin(y)}{x - y}, \quad g(x,y) = \frac{1 - \cos(\sqrt{x^2+y^2})}{x^2+y^2}.$$

Exercise 07:

Consider the following function:

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

1) Calculate $f''_{xy}(0,0)$ and $f''_{yx}(0,0)$.

2) Are the functions f''_{xy} and f''_{yx} continuous at $(0,0)$?

Extra Exercises

Exercise 08:

Study the existence of local extrema of the following functions:

$f_1(x,y) = x^2-xy+y^2+3x-2y+1$, $f_2(x,y) = x^3+2xy+y^2-1$, $f_3(x,y) = 3xy-x^3-y^3$, $f_4(x,y) = x^2y^2(1+x+$

Exercise 09:

Study the existence of local and global extrema of the following functions on their domains of definition:

$$f(x,y) = \sin(x) + y^2 + y + 2, \quad g(x,y) = x^3 - 3x(1+y^2).$$

Exercise 10:

Provide the second-order Taylor expansion of the following functions around (x_0, y_0)*:*

$$f(x,y) = x^2 + xy + 4y^2 \quad \textit{at} \quad (x_0, y_0) = (1,2),$$

$$g(x,y) = x\ln(y) + y\ln(x) \quad \textit{at} \quad (x_0, y_0) = (1,1),$$

$$h(x,y) = \ln(1 + 2x + 2y) \quad \textit{at} \quad (x_0, y_0) = (0,0).$$

4.9 Solution:

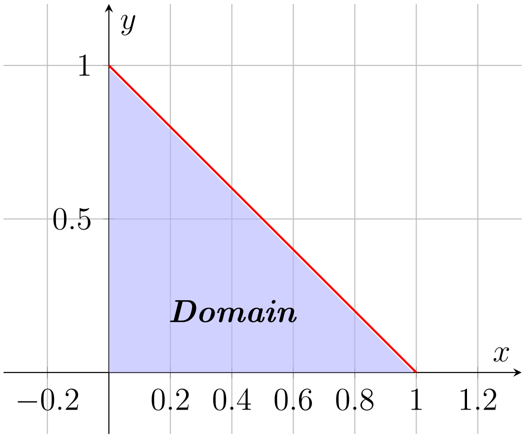
Solution to exercise 1

***1.** *Domain of* $f_1(x,y) = \dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{y}} + \ln(1-x-y)$

We need:

$$x > 0, \quad y > 0, \quad x + y < 1$$

Domain: $x > 0$, $y > 0$, $x + y < 1$



***2.** *Domain of* $f_2(x,y) = \dfrac{1}{\sqrt{x^2+y^2-1}} + \sqrt{4-x^2-y^2}$

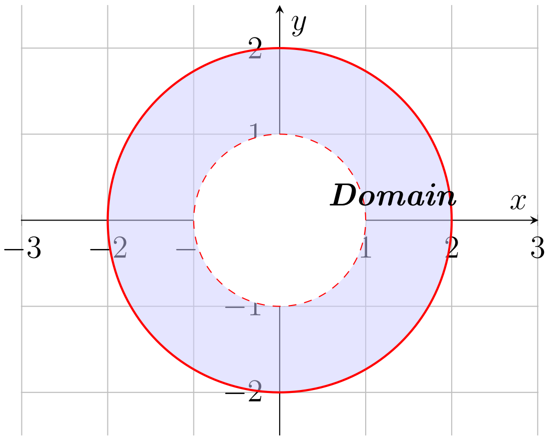
We need:

$$x^2 + y^2 - 1 > 0 \quad \text{and} \quad 4 - x^2 - y^2 \geqslant 0$$

$$\Rightarrow \quad 1 < x^2 + y^2 \leqslant 4.$$

*So the domain is the **annulus** (ring-shaped region) between the circles of radius 1 and 2.*

D_3: $1 < x^2 + y^2 \leqslant 4$



Solution to exercise of 2

1)

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4+y^4} = \lim_{r\to 0^+} \frac{r^3\cos^3(\theta)\sin(\theta)}{r^4(\cos^4(\theta)+\sin^4(\theta))} = \frac{\cos^3(\theta)\sin(\theta)}{\cos^4(\theta)+\sin^4(\theta)}$$

The limit depends on θ, hence it does not exist.

2)

$$\lim_{(x,y)\to(0,0)} \frac{e^{x+y}-1}{x} = \lim_{(x,y)\to(0,0)} \frac{x+y}{x} = \lim_{r\to 0^+} \frac{r\cos(\theta)+r\sin(\theta)}{r\cos(\theta)} = \frac{\cos(\theta)+\sin(\theta)}{\cos(\theta)}$$

The limit depends on θ, hence it does not exist.

3)

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{|x|+|y|} = \lim_{(x,y)\to(0,0)} \frac{xy}{|x|+|y|} = 0$$

4)

$$\lim_{(x,y)\to(0,0)} (x+y)\sin\left(\frac{1}{x^2+y^2}\right)$$

We use the inequality:

$$\left|\sin\left(\frac{1}{x^2+y^2}\right)\right| \leqslant 1,$$

so

$$\left|(x+y)\sin\left(\frac{1}{x^2+y^2}\right)\right| \leqslant |x+y| \to 0 when (x,y) \to (0,0).$$

We conclude by the Squeeze Theorem:

$$\lim_{(x,y)\to(0,0)} (x+y)\sin\left(\frac{1}{x^2+y^2}\right) = 0.$$

5)

$$\lim_{(x,y)\to(0,0)} \frac{|x|+|y|}{x^2+y^2} = \lim_{r\to 0} \frac{r(\cos(\theta)+\sin(\theta))}{r^2} = \lim_{r\to 0} \frac{\cos(\theta)+\sin(\theta)}{r} = +\infty$$

6)

$$\lim_{(x,y)\to(0,a)} \frac{\sin(xy)}{y} = \lim_{(x,y)\to(0,0)} \frac{xy}{y} = \lim_{(x,y)\to(0,0)} x = 0$$

7) For $y = 2x$, we get $\lim_{x\to 0} \frac{-2x^3}{3} = 0$.
For $y = x + x^4$, we get $\lim_{x\to 0} \frac{1+x^3}{-x^3-2} = -\frac{1}{2}$.

Solution to Exercise 3

1) Domain of $f_1(x,y) = \frac{xy^2}{x^2+y^4}$ is $\mathbb{R}^2$.
f_1 is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ because it is a ratio of polynomials.

At $(0,0)$: Consider two paths:

1. *For $y = x$, we get $\lim_{x\to 0} \frac{x}{1+x^2} = 0$.*
2. *For $y = x^2$, we get $\lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2} \neq f_1(0,0)$.*
   Therefore, f_1 is not continuous at $(0,0)$.

2) Domain of $f_2(x,y) = y^2 \sin\left(\frac{x}{y}\right)$ is $\mathbb{R}^2$.
f_2 is continuous on $\mathbb{R}^2 \setminus \{(a,0) \mid a \in \mathbb{R}\}$ because it is the product of a continuous function (y^2) and a bounded function ($\sin\left(\frac{x}{y}\right)$).
$\lim_{(x,y)\to(0,0)} y^2 \sin\left(\frac{x}{y}\right) = 0 \neq 0$ (since the function $(x,y) \to \sin\left(\frac{x}{y}\right)$ is bounded and $(x,y) \to y^2$ tends to 0).
Therefore, f_2 is not continuous at $(0,0)$.

3) $f_3(x,y) = \begin{cases} 2x^2+y^2-1 & \textit{if } x^2+y^2 > 1 \\ x^2 & \textit{otherwise} \end{cases}$

f_3 is defined on $\mathbb{R}^2$.

f_3 is continuous on $\mathbb{R}^2 \setminus \{(x,y) \mid x^2+y^2 = 1\}$ because it is a polynomial function on these sets.

Now, verify continuity on $x^2 + y^2 = 1$:

Let (x_n, y_n) be a sequence converging to a point (a, b) such that $a^2 + b^2 = 1$.

1. *If $x_n^2 + y_n^2 > 0$, then $f_3(x_n, y_n) = 2x_n^2 + y_n^2 - 1$.*
2. *If $x_n^2 + y_n^2 \leqslant 0$, then $f_3(x_n, y_n) = x_n^2$.*
   As $2x_n^2 + y_n^2 - 1 \to 2a^2 + b^2 - 1 = a^2$ and $x_n^2 \to a^2$, and $a^2 + b^2 = 1$, both limits are equal. Therefore, f_3 is continuous at (a, b).

Solution to Exercise 4

1) $f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Calculate $f'_x(0,0)$ and $f'_y(0,0)$:

$$f'_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}}{h} = 1$$

$$f'_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{-h^3}{h^2}}{h} = -1.$$

2) Assume by contradiction that f is differentiable at $(0,0)$:
Then $df(0,0)(h_1,h_2) = f'_x(0,0)h_1 + f'_y(0,0)h_2 = h_1 - h_2$.
So,

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2) - f(0,0) - df(0,0)(h_1,h_2)}{\sqrt{h_1^2 + h_2^2}} = 0$$

Calculate this limit in polar coordinates:

$$\lim_{r\to 0} \frac{\frac{r^3\cos^3(\theta) - r^3\sin^3(\theta))}{r^2} - rcos(\theta) + rsin(\theta)}{r} = \lim_{r\to 0}(\cos^3(\theta) + \sin^3(\theta)) - cos(\theta) + sin(\theta)$$

This limit depends on θ, so it does not exist.

3) If f'_x and f'_y are continuous at $(0,0)$, then f is differentiable at $(0,0)$. This contradicts part 2.

Solution to Exercise 5

1) $f(x,y) = \begin{cases} \frac{xy}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ because it is a ratio of two polynomials.

At $(0,0)$:

For the path $x = y$:

$$\lim_{x\to 0} \frac{x^2}{x^2+x^4} = \lim_{x\to 0} \frac{1}{1+x^2} = 1$$

Thus, $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$.

Therefore, f is not continuous at $(0,0)$.

2) Calculate $f'_x(0,0)$ and $f'_y(0,0)$:

$$f'_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h\cdot 0}{h^2}}{h} = 0.$$

$$f'_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0\cdot h}{h^2}}{h} = 0.$$

The partial derivatives exist despite the discontinuity of the function.

Solution to Exercise 6

1) $f(x,y) = \frac{\sin(x)-\sin(y)}{x-y}$ is defined on $\mathbb{R}^2 \setminus \{(x,y) \mid x = y\}$. We investigate the limit $\lim_{(x,y)\to(a,a)} f(x,y)$ where $a \in \mathbb{R}$. By the definition of the derivative, we have:

$$\lim_{(x,y)\to(a,a)} \frac{\sin(x)-\sin(y)}{x-y} = \cos(a) \in \mathbb{R}$$

Therefore, f can be extended by continuity to the set $\{(x,y) \mid x = y\}$. The extended function $\tilde{f}(x,y)$ is defined as:

$$\tilde{f}(x,y) = \begin{cases} \frac{\sin(x)-\sin(y)}{x-y} & \text{if } x \neq y \\ \cos(a) & \text{if } x = y \end{cases}$$

This extension ensures continuity of f on $\mathbb{R}^2$.

Solution to Exercise 7

1) To compute second-order partial derivatives, we first calculate first-order derivatives:

$$\frac{\partial f}{\partial x} = \frac{y(x^4 - 3x^2y^2 - y^4)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x(-x^4 + 3x^2y^2 + y^4)}{(x^2+y^2)^2}$$

Also, the partial derivatives are:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 0$$

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

And the mixed partial derivatives are:

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \lim_{h\to 0} \frac{\frac{\partial f}{\partial y}(h,0) - \frac{\partial f}{\partial y}(0,0)}{h} = 1$$

$$\frac{\partial^2 f(x,y)}{\partial y \partial x} = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = -1$$

2) The cross derivatives are different at $(0,0)$ and f is anti-symmetric, according to the Schwartz theorem, $\frac{\partial^2 f(x,y)}{\partial x \partial y}$ and $\frac{\partial^2 f(x,y)}{\partial y \partial x}$ are not continuous in$(0,0)$.

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